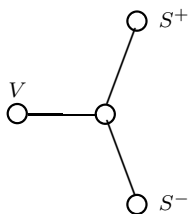


Trigonal curves and Galois $\mathrm{Spin}(8)$ -bundles

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The group $\mathrm{Spin}(8)$ occupies a special position among the complex simple Lie groups, in having outer automorphism group S_3 and an appealing Dynkin diagram:



The outer nodes are dual to three fundamental 8-dimensional representations, the standard representation V on which the group acts via the double cover $\mathrm{Spin}(8) \rightarrow \mathrm{SO}(8)$, and the half-spinor representations S^\pm . These spaces are permuted by the action of S_3 , and between any two there is a Clifford multiplication to the third. It is possible to identify all three spaces with the complex Cayley algebra \mathbb{O} , and Clifford multiplication with multiplication of octonions. In this sense $\mathrm{Spin}(8)$ is a member of the ‘exceptional’ club, and is closely related to the other exceptional groups $G_2 = \mathrm{Aut} \mathbb{O}$, F_4 , E_6 etc.

In this paper we study the moduli of principal holomorphic $\mathrm{Spin}(8)$ -bundles over an algebraic curve. However, in order to exploit triality, and to obtain a moduli space with particularly nice properties, we impose some additional constraints on our bundles. We suppose that X is an algebraic curve with an S_3 -action. The group S_3 then acts in two ways on $\mathrm{Spin}(8)$ -bundles over a curve: by pull-back under the action on X , and by the triality action on the structure group. We call a $\mathrm{Spin}(8)$ -bundle *Galois* if these two actions coincide, that is, if it is a fixed point of the group action

$$F \mapsto u^* F^u, \quad u \in S_3,$$

on the moduli variety $\mathcal{M}_X(\mathrm{Spin}(8))$ of (semistable) $\mathrm{Spin}(8)$ -bundles. We denote the fixed-point set by $\mathcal{F}_X \subset \mathcal{M}_X(\mathrm{Spin}(8))$, and distinguish the subvariety of \mathcal{F}_X consisting of bundles which admit a lift of the S_3 -action (see Lemma 2.2.2 for the precise notion). This subvariety has a partial desingularisation \mathcal{N}_X parametrising pairs (F, Λ) where $F \in \mathcal{F}_X$ and Λ is an S_3 -lift, or more precisely a splitting $\Lambda : S_3 \rightarrow \mathcal{G}_F$ of the Mumford group associated to F (see Definition 2.2.3). \mathcal{N}_X is the main object of interest of this paper.

In §3 we compute the tangent spaces at stable points and the semistable boundary of \mathcal{N}_X (see 3.1.3 and 3.2.1). The best situation, to which we mainly restrict ourselves, is where the

quotient X/S_3 is isomorphic to \mathbb{P}^1 , and in this case we show that \mathcal{N}_X is smooth at stable Galois bundles and that the locus of nonstable bundles can be identified with the moduli space $\mathcal{SU}_{X/\sigma}(2)$ of rank 2, trivial determinant vector bundles on the quotient of X by the involution $\sigma \in S_3$ which exchanges the spinor representations (see 1.1.1 for notation). Both properties are in marked contrast to $\mathcal{M}_X(\text{Spin}(8))$ itself, whose semistable boundary is more complicated to describe and whose stable points are not necessarily smooth but may have finite quotient singularities.

The condition $X/S_3 \cong \mathbb{P}^1$ just means that $X = \mathcal{G}(C)$ is the Galois closure over \mathbb{P}^1 of the trigonal curve $C = X/\sigma$, and conversely one can construct \mathcal{N}_X starting from any (non-cyclic) trigonal curve $C \xrightarrow{3:1} \mathbb{P}^1$. Indeed, this was precisely our motivation for the construction, and we shift attention to the trigonal point of view in §4. The curve $X = \mathcal{G}(C)$ is a branched double cover of C ; we write $\mathcal{N}_C = \mathcal{N}_X$; and the main result of the paper is the following:

Theorem 4.2.2. *Given a (non-cyclic) trigonal curve $C \xrightarrow{3:1} \mathbb{P}^1$ there exists a projective moduli space \mathcal{N}_C (parametrising Galois $\text{Spin}(8)$ -bundles on $\mathcal{G}(C)$) which admits an inclusion $\mathcal{SU}_C(2) \hookrightarrow \mathcal{N}_C$ such that \mathcal{N}_C is smooth and of dimension $7g - 14$ away from the image of $\mathcal{SU}_C(2)$.*

(Cyclic trigonal curves need to be treated separately as the Galois group is $\mathbb{Z}/3$ rather than S_3 , and although many of the computations work well here—and we expect that the above theorem remains true—we confine ourselves to brief remarks on this case at various points—see the end of §2.3.)

The moduli space $\mathcal{SU}_C(2)$ rank 2 semistable vector bundles with trivial determinant is an object of considerable interest. As well as containing the Jacobian Kummer as its singular locus, a fundamental geometric feature is the so-called *Schottky configuration* of Prym Kummers. (See for example [7], [12].) That is, the group $J_C[2]$ of 2-torsion points in the Jacobian acts on $\mathcal{SU}_C(2)$ by tensor product; the fixed-point set of an element $\eta \in J_C[2]$ is a pair of (isomorphic) Kummer varieties $(P_\eta/\pm) \cup (P_\eta^-/\pm)$ of dimension $g - 1$. P_η is the Prym variety of the unramified double cover $C_\eta \rightarrow C$ corresponding to η , and the map $P_\eta \cup P_\eta^- \rightarrow \mathcal{SU}_C(2)$ is (up to a choice of $\eta^{\frac{1}{2}}$) the direct image of line bundles from C_η . The incidence relations among these Kummer varieties as η varies, interpreted via the embedding $\mathcal{SU}_C(2) \hookrightarrow |2\Theta|$ (where Θ is the Riemann theta divisor in the Jacobian) correspond precisely to the Schottky-Jung-Donagi identities among the thetanulls of J_C and the Pryms.

It turns out that when C is trigonal the Schottky configuration too has a ‘fattening’ in the moduli space \mathcal{N}_C . By a beautiful and well-known construction of Recillas [18], each trigonal Prym P_η is isomorphic as a principally polarised abelian variety to the Jacobian of a tetragonal curve $R_\eta \xrightarrow{4:1} \mathbb{P}^1$. Thus a trigonal Schottky configuration consists of Jacobian Kummers J_{R_η}/\pm . Each of these is also the singular locus of a moduli space of bundles $\mathcal{SU}_{R_\eta}(2)$. We shall show:

Theorem 5.4.3. *Given a trigonal curve C and a nonzero 2-torsion point $\eta \in J_C[2]$ there exists*

(up to a choice of $\eta^{\frac{1}{2}}$) a natural map $\mathcal{SU}_{R_\eta}(2) \hookrightarrow \mathcal{N}_C$ for which the following diagram commutes:

$$\begin{array}{ccc} P_\eta & \xrightarrow{\sim} J_{R_\eta} \rightarrow & \mathcal{SU}_{R_\eta}(2) \\ \downarrow & & \downarrow \\ \mathcal{SU}_C(2) & \hookrightarrow & \mathcal{N}_C. \end{array}$$

One can view the right-hand side of this diagram, as η varies, as a nonabelian Schottky configuration singular along the classical Schottky configuration on the left-hand side.

Finally, we wish to add a word about the motivation for these constructions. Originally, this was our interest the projective embedding (see [2], [6]; we assume C is nonhyperelliptic unless $g = 2$)

$$\phi : \mathcal{SU}_C(2) \rightarrow |2\Theta| = \mathbb{P}^{2^g-1}.$$

It is well-known that for $g = 2$ the map ϕ is an isomorphism $\mathcal{SU}_C(2) \xrightarrow{\sim} \mathbb{P}^3$ (see [11]); and that for $g = 3$ (see [13]) the image $\phi(\mathcal{SU}_C(2)) \subset \mathbb{P}^7$ is the unique Heisenberg-invariant quartic—the *Coble quartic*—singular along the Kummer variety $J_C/\pm \subset |2\Theta|$. When $g = 4$ it was shown in [15] that in $|2\Theta| = \mathbb{P}^{15}$ there exists a unique Heisenberg-invariant quartic singular along the image $\phi(\mathcal{SU}_C(2))$. It was shown, moreover, that the fixed-point set of $\eta \in J_C[2]$ in this quartic is a pair of Coble quartics $\mathcal{SU}_{R_\eta}(2)$ of the corresponding Recillas curve. However, the questions remained open: what is this 14-dimensional quartic as a moduli space?, and is its singular locus equal to $\mathcal{SU}_C(2)$?

The crucial point here is that a general curve of genus 4 is trigonal (in two ways. However, by Torelli's theorem R_η is independent of the choice of trigonal structure). In view of the results of the present paper it is natural to expect that the genus 4 quartic in \mathbb{P}^{15} is \mathcal{N}_C . We hope to pursue this question in a sequel.

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1 Spin(8) and triality

We begin by recalling the triality story for Spin(8) from various points of view. Roughly, §1.1 will be used in the discussion of Galois bundles of section 2; §1.2 and §1.3 are needed for the discussion of stability in section 3; and §1.4 will be used in the computation of the dimension of the moduli space in section 4.

1.1 The group Spin(8)

We shall always denote by $V \cong \mathbb{C}^{2n}$ the standard orthogonal representation of Spin(2n), and by $S^\pm \cong \mathbb{C}^{2^{n-1}}$ its half-spinor representations. We shall denote by q the quadratic form on V ;

if we fix a decomposition $V = N \oplus N^\vee$, where N and N^\vee are maximal isotropic subspaces, dual via q , then by definition

$$(1) \quad S^+ = \bigwedge^{\text{even}} N, \quad S^- = \bigwedge^{\text{odd}} N.$$

We need to recall some facts about S^\pm (see [4]). Start with the bilinear pairing

$$\begin{aligned} r : \bigwedge N \otimes \bigwedge N &\rightarrow \bigwedge^n N \cong \mathbb{C} \\ s \otimes t &\mapsto (\beta(s) \wedge t)_n \end{aligned}$$

where β is the principal anti-involution of $\bigwedge N$, i.e. it is the identity on N and reverses multiplication; and where $()_n$ denotes the component of top degree. When n is odd this form vanishes on each of S^\pm and induces a nondegenerate pairing $r : S^+ \otimes S^- \rightarrow \mathbb{C}$. When n is even it restricts to a nondegenerate pairing on each of S^\pm , which is symmetric precisely when $n \equiv 0(4)$.

We shall be concerned with the case $n = 4$; here the pairing r determines quadratic forms q^\pm on the spinor spaces S^\pm . Moreover, these determine an embedding (in fact the projection into the product of any two factors is still an embedding)

$$(2) \quad \text{Spin}(8) \hookrightarrow SO(V) \times SO(S^+) \times SO(S^-).$$

There is a well-known triality relationship among the three orthogonal factors here. $\text{Spin}(8)$ has outer automorphism group S_3 , and we shall recall explicitly how S_3 acts on $\text{Spin}(8)$, or more precisely how to split the sequence

$$1 \rightarrow \text{Inn Spin}(8) \rightarrow \text{Aut Spin}(8) \rightarrow S_3 \rightarrow 1.$$

1.1.1 Notation. We shall use, here and in later sections, generators $\sigma = (23)$ and $\tau = (123) \in S_3$; thus σ should interchange S^\pm , as an involution of $\text{Spin}(8)$; while $\sigma\tau : V \leftrightarrow S^+$ and $\sigma\tau^2 : V \leftrightarrow S^-$.

Choose unit vectors $v_0 \in V$, $s_0 \in S^+$ and $t_0 = v_0 \cdot s_0 \in S^-$. Given these choices we let S_3 act on the vector space $V \oplus S^+ \oplus S^-$ as follows. σ acts by reflection of V in v_0^\perp and by Clifford multiplication $v_0 : S^+ \leftrightarrow S^-$. Similarly the group elements $\sigma\tau$ and $\sigma\tau^2$ are represented by s_0 and t_0 respectively.

Given these choices we shall denote the resulting representation by

$$\rho : S_3 \rightarrow O(V \oplus S^+ \oplus S^-).$$

This intertwines an S_3 -action on $\text{Spin}(8)$ by $g \mapsto g^u := \rho(u)^{-1}g\rho(u)$ for $u \in S_3$ and $g \in \text{Spin}(8) \subset O(V \oplus S^+ \oplus S^-)$. Note that the subgroup of $O(V \oplus S^+ \oplus S^-)$ generated by the images of S_3 and $\text{Spin}(8)$ is isomorphic to the semi-direct product $\text{Spin}(8) \rtimes S_3$. We note for later use that the group law on $\text{Spin}(8) \rtimes S_3$ is

$$(3) \quad (g, u)(h, v) = (gh^u, uv) \quad u, v \in S_3 \quad g, h \in \text{Spin}(8).$$

In view of triality, there is a multiplication map $S^+ \otimes S^- \rightarrow V$ permuted by S_3 with the Clifford multiplications $V \otimes S^\pm \rightarrow S^\mp$. This can be described in terms of a trilinear form:

$$(4) \quad \begin{aligned} c : V \otimes S^+ \otimes S^- &\rightarrow \mathbb{C} \\ v \otimes s \otimes t &\mapsto q^-(v \cdot s, t) = q^+(v \cdot t, s) \stackrel{\text{def}}{=} q(s \cdot t, v). \end{aligned}$$

The first equation is easily checked; the second defines the Clifford multiplication by:

$$\begin{array}{ccc} S^+ \otimes S^- & \xrightarrow{c} & V^\vee \\ & \searrow & \downarrow q \\ & & V \end{array}$$

1.1.2 *Remark.* One can use the three Clifford multiplications to define a commutative algebra structure on $V \oplus S^+ \oplus S^-$ by taking the product of two vectors in the same summand to be zero. The resulting algebra, called the *Chevalley algebra*, is an example of a vertex operator algebra. (See [5].)

1.1.3 *LEMMA.* *The orthogonal action $\rho : S_3 \rightarrow O(V \oplus S^+ \oplus S^-)$ preserves the cubic form induced by the trilinear form c , and hence also the Chevalley algebra structure, and determines an S_3 -action on $\text{Spin}(8)$ as above. Moreover, every splitting $S_3 \rightarrow \text{Aut Spin}(8)$ arises in this way.*

1.1.4 *Remark.* Given the above choices V acquires an algebra structure

$$\begin{aligned} V \times V &\rightarrow S^+ \times S^- \xrightarrow{\cdot} V \\ (u, v) &\mapsto (t_0 \cdot u, s_0 \cdot v) \end{aligned}$$

It is well-known (and will follow from Remark 1.2.2 below) that this is precisely the (complex) Cayley algebra $V \cong \mathbb{O}$ with centre $\langle v_0 \rangle$. One deduces from this fact a characterisation of the image of (2):

$$(5) \quad \text{Spin}(8) = \{(a, b, c) \in SO(\mathbb{O}) \times SO(\mathbb{O}) \times SO(\mathbb{O}) \mid a(u)b(v) = c(uv) \quad \forall u, v \in \mathbb{O}\}.$$

In this language the triality action is given by $\sigma : (a, b, c) \mapsto (a', c', b')$ and $\tau : (a, b, c) \mapsto (b', c, a')$ where $a'(u) = \overline{a(\overline{u})}$.

1.1.5 *LEMMA.* *For an isotropic subspace $U \subset V$ and an involution $g \in S_3$ not preserving V , consider the Clifford multiplication maps:*

$$\mu : U \otimes U \xrightarrow{\text{id} \otimes g} V \otimes S^\pm \xrightarrow{\cdot} S^\mp$$

If $\dim U = 1$ then $\mu = 0$, and if $\dim U = 2$ then $\text{rank } \mu \leq 1$.

Proof. Suppose first that $\dim U = 1$. Identifying V, S^+, S^- with \mathbb{O} as in Remark 1.1.4 the map μ becomes $x \otimes x \mapsto x\bar{x} = \|x\| = 0$. (See also Remark 1.2.2). If $\dim U = 2$ then the symmetric tensors $S^2 U \subset U \otimes U$ are spanned by squares, and so are contained in kernel of μ . This shows that $\text{rank } \mu \leq 1$. \square

Finally, note that the centre of $\text{Spin}(8)$ is $Z(\text{Spin}(8)) = \mathbb{Z}/2 \times \mathbb{Z}/2$, consisting of the matrices of 8×8 blocks:

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

1.2 Triality in terms of 2×2 matrices

We can always split $V = \mathbb{C}^8$, as an orthogonal space, into a direct sum of orthogonal \mathbb{C}^4 s. In turn \mathbb{C}^4 can be identified with $\text{Hom}(A, B)$ where $A, B \cong \mathbb{C}^2$, with orthogonal structure given by the determinant (geometrically, every smooth complex quadric surface is the Segre $\mathbb{P}^1 \times \mathbb{P}^1$). The spaces A^\vee and B are then the spinor representations, where $\text{Spin}(4) = SL(2) \times SL(2)$. We can view this as saying that A, B carry fixed (complex) orientations $\lambda_A : \bigwedge^2 A \xrightarrow{\sim} \mathbb{C}$ and $\lambda_B : \bigwedge^2 B \xrightarrow{\sim} \mathbb{C}$.

Let us generalise this situation for a moment. Suppose that $A, B \cong \mathbb{C}^n$ and are each equipped with a fixed orientation. Then any $u \in \text{Hom}(A, B)$ has an *adjugate* homomorphism $\bar{u} \in \text{Hom}(B, A)$. This is just the transpose of the composition

$$A^\vee \xrightarrow{\lambda_A} \bigwedge^{n-1} A \xrightarrow{\bigwedge^{n-1} u} \bigwedge^{n-1} B \xrightarrow{\lambda_B} B^\vee,$$

and with respect to chosen bases is given by the transpose matrix of signed cofactors of the matrix representing u . Adjugacy is therefore a natural birational involution $\text{Hom}(A, B) \leftrightarrow \text{Hom}(B, A)$ for oriented vector spaces; when $n = 2$ it is a linear involution. (The reader may care to consider the next case $n = 3$: projectively the map blows up the Segre $\mathbb{P}^2 \times \mathbb{P}^2$ and contracts secant lines down to the dual Segre fourfold.)

In the case $n = 2$ the determinant of a map between oriented spaces can be interpreted as a quadratic form $\det : \text{Hom}(A, B) \rightarrow \mathbb{C}$ whose polarisation is the symmetric bilinear form

$$\langle u, v \rangle = \frac{1}{2} \text{trace } u\bar{v}, \quad u, v \in \text{Hom}(A, B).$$

We now return to our 8-dimensional orthogonal space, which we shall decompose as

$$V = \text{Hom}(A, B) \oplus \text{Hom}(C, D)$$

where A, B, C, D are oriented \mathbb{C}^2 s. It follows from Lemma 2.1.1, and the identifications $A \xrightarrow{\sim} A^\vee$ etc given by the orientations, that the spinor spaces are

$$\begin{aligned} S^+ &= \text{Hom}(B, C) \oplus \text{Hom}(A, D), \\ S^- &= \text{Hom}(C, A) \oplus \text{Hom}(B, D). \end{aligned}$$

It is straightforward to calculate the Clifford multiplication maps: we shall take on V the quadratic form $(a, b) \mapsto \det a - \det b$. One then finds:

$$\begin{aligned} V \otimes S^+ &\rightarrow S^- \\ (a, b) \otimes (x, y) &\mapsto (\bar{a} \bar{x} + \bar{y} b, bx + y \bar{a}) \end{aligned}$$

with similar expressions for the other two maps. (The rule is: for each of the two Hom summands, simply add the only possible composites, allowing adjugates, so that the expression is well-defined.)

1.2.1 Remark. The orientations on the spaces A, B, C, D induce identifications $\bigwedge^8 V \cong \bigwedge^8 S^+ \cong \bigwedge^8 S^-$, so the determinant of each Clifford multiplication map is a well-defined scalar. For each pair of 2×2 matrices $(a, b) \in V = \text{Hom}(A, B) \oplus \text{Hom}(C, D)$ one finds that the linear map $m_{a,b} : S^+ \rightarrow S^-$ satisfies:

$$\det m_{a,b} = -(\det a - \det b)^4.$$

This corresponds to the fact that the rank drops by 4 if $\det a = \det b$, i.e. if $(a, b) \in V$ is isotropic. One can check this directly (e.g. using Maple) by Gauss-Jordan elimination.

1.2.2 Remark. If we fix unimodular isomorphisms $A \xrightarrow{\sim} B \xrightarrow{\sim} C \xrightarrow{\sim} D$ we obtain a natural choice of unit vectors $v_0 = (1, 0) \in V$, $s_0 = (1, 0) \in S^+$, $t_0 = (1, 0) \in S^-$. (Note that $t_0 = v_0 s_0$ under the Clifford multiplication above.) With this choice the algebra structure of Remark 1.1.4 becomes:

$$\begin{aligned} V \times V &\rightarrow V \\ (a, b)(c, d) &\mapsto (ac + \bar{d}b, da + b\bar{c}). \end{aligned}$$

On the other hand, the chosen isomorphisms identify $V = \text{End } A \oplus \text{End } A$ where $\text{End } A \cong \text{Mat}_2(\mathbb{C}) \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, the complexification of the quaternions. Moreover, quaternionic conjugation is precisely the map which sends a matrix to its adjugate. It follows that the above algebra structure on V is exactly the (split) Cayley-Dickson process applied to $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ (see [8] pp.105–106), i.e. identifies V with \mathbb{O} .

1.3 6-dimensional quadrics

Let $Q \subset V$ and $Q^\pm \subset S^\pm$ be the quadrics defined by the respective quadratic forms on the basic representations of $\text{Spin}(8)$. Each quadric has two families of 4-dimensional isotropic spaces, which we shall sometimes refer to as α -planes and β -planes. Such subspaces $A, B \subset Q$ belong to opposite families if and only if

$$\dim A \cap B \equiv 1 \pmod{2}.$$

Moreover, these families are parametrised precisely by the quadrics Q^\pm as follows.

Given vectors $s \in S^+$ and $t \in S^-$, consider the Clifford multiplication maps $m_s : V \leftrightarrow S^-$ and $m_t : V \leftrightarrow S^+$. We have observed in Remark 1.2.1 that these maps have nonzero kernel exactly when s, t respectively are isotropic, and that then the rank is 4. We define:

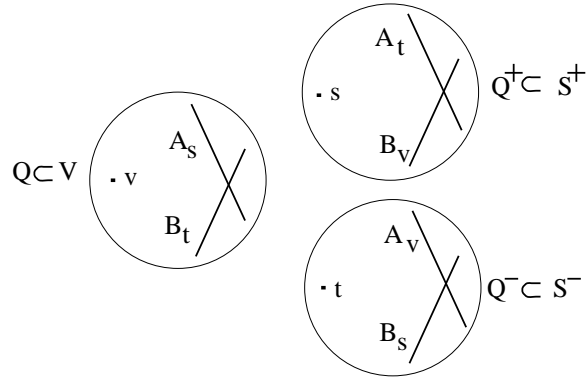
$$\begin{aligned} A_s &= \ker \{V \xrightarrow{m_s} S^-\} = \text{Im} \{S^- \xrightarrow{m_s} V\} \subset Q \subset V, \\ B_t &= \ker \{V \xrightarrow{m_t} S^+\} = \text{Im} \{S^+ \xrightarrow{m_t} V\} \subset Q \subset V. \end{aligned}$$

One readily checks (see [4], or use the set-up of §1.2) these equalities, that they define isotropic subspaces, and that:

$$\dim A_s \cap B_t = \begin{cases} 1 & \text{spanned by } s \cdot t \in V \text{ if } s \cdot t \neq 0, \\ 3 & \text{if } s \cdot t = 0. \end{cases}$$

In particular A_s, B_t are in opposite families.

By triality (i.e. using (4) in §1) we can make the same constructions in each of S^\pm . We summarise our notation in the following diagram:



(6)

Note that:

$$(7) \quad s \cdot t = 0 \iff s \in A_t \iff t \in B_s \iff \dim A_s \cap B_t = 3.$$

Moreover, there are canonical dualities $A_s = B_s^\vee$ for $s \in Q^\pm$ or Q ; this follows, for example, via the quadratic form on V , from the exact sequences given by Clifford multiplication:

$$(8) \quad \begin{aligned} 0 &\rightarrow A_s \rightarrow V \rightarrow B_s \rightarrow 0, \\ 0 &\rightarrow B_t \rightarrow V \rightarrow A_t \rightarrow 0. \end{aligned}$$

Geometrically, of course, A_t is the set of $s \in Q^+$ for which $\dim A_s \cap B_t = 3$, and so parametrises hyperplanes in B_t .

Similarly, for distinct $s, s' \in Q^+$ we have

$$(9) \quad \dim A_s \cap A_{s'} = \begin{cases} 2 & \text{if } \langle s, s' \rangle = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this fact identifies the Grassmannian $\text{Grass}(2, A_s)$ with the 4-dimensional (tangent) quadric in $\mathbb{P}(s^\perp/s)$.

We can generalise the above correspondence by defining, for any isotropic subspace $U \subset V$, and using the trilinear form (4):

$$\begin{aligned} A_U &= \ker \{S^- \xrightarrow{c} (U \otimes S^+)^\vee\} = \text{Im} \{U \otimes S^+ \xrightarrow{c} (S^-)^\vee\}^\perp, \\ B_U &= \ker \{S^+ \xrightarrow{c} (U \otimes S^-)^\vee\} = \text{Im} \{U \otimes S^- \xrightarrow{c} (S^+)^\vee\}^\perp. \end{aligned}$$

Note that $A_U = \bigcap_{v \in U} A_v \subset Q^-$ unless U is an α -plane A_s , in which case $A_U = B_s$ —with similar remarks holding for B_U . If $\dim U = 1, 2, 3$ then $\dim A_U = \dim B_U = 4, 2, 1$ respectively; if $U = B_t$ is a β -plane then A_U is 1-dimensional and is spanned by $t \in S^-$.

By triality one can now extend the notation of diagram (6) to arbitrary *isotropic subspaces* v, s, t . The following properties are essentially tautological:

1.3.1 LEMMA. *For any isotropic subspace $U \subset V$, S^+ or S^- , with $\dim U \neq 3$, we have:*

- (i) $A_{B_U} = B_{A_U} = U$;
- (ii) $A_{A_U} = B_U$ and $B_{B_U} = A_U$.

1.3.2 LEMMA. *Suppose that $U, R \subset V \oplus S^+ \oplus S^-$ are 2-dimensional isotropic subspaces contained in different summands, and that Clifford multiplication vanishes on $U \otimes R$. Then either $U = A_R$, $R = B_U$ or $U = B_R$, $R = A_U$.*

Finally, fix a triality action $\rho : S_3 \rightarrow O(V \oplus S^+ \oplus S^-)$ as in §1.1.

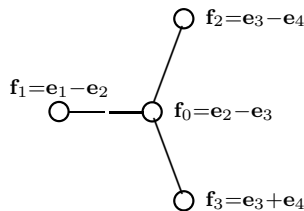
1.3.3 LEMMA. *For any $g \in S_3$ and isotropic subspace $U \subset V$, S^+ or S^- we have:*

$$gA_U = \begin{cases} A_{gU} \\ B_{gU} \end{cases} \quad gB_U = \begin{cases} B_{gU} & \text{if } \text{sgn } g = +1, \\ A_{gU} & \text{if } \text{sgn } g = -1. \end{cases}$$

Proof. Suppose that $U \subset V$ (the argument for $U \subset S^\pm$ is the same). Then $A_U = \{t \in S^- | c(u, s, t) = 0 \forall s \in S^+, u \in U\}$ using the trilinear form (4). Since this form is S_3 -invariant gA_U consists of $x = gt \in V, S^+$ or S^- (depending on g) such that $c(gu, gs, x) = 0$ for all $u \in U$, $s \in S^+$ (where we temporarily disregard the order of the arguments of c), i.e. $c(u', s', x) = 0$ for all $u' \in gU$ and $s' \in gS^+$. This shows that $gA_U = A_{gU}$ or B_{gU} ; the assertion that which one depends on $\text{sgn } g$ follows easily from diagram (6). \square

1.4 Triality action on the Lie algebra

We need next to understand how the Lie algebra $\mathfrak{g} = \mathfrak{so}(8)$ decomposes under the action of S_3 . We write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$ where \mathfrak{h} is a Cartan subalgebra with orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ and \mathfrak{g}_\pm are spanned by the positive and negative root spaces with respect to simple roots:



The 24 roots of \mathfrak{g} are $\pm \mathbf{e}_i \pm \mathbf{e}_j$. The group S_3 acts on the Dynkin diagram, and its action on \mathfrak{g} is obtained by examining the action on the root spaces. This can be made transparent by rewriting the 12 positive roots as:

$$\begin{array}{cccc} \mathbf{f}_1, & \mathbf{f}_0 + \mathbf{f}_1, & \mathbf{f}_0 + \mathbf{f}_2 + \mathbf{f}_3, & \mathbf{f}_0, \\ \mathbf{f}_2, & \mathbf{f}_0 + \mathbf{f}_2, & \mathbf{f}_0 + \mathbf{f}_1 + \mathbf{f}_3, & \mathbf{f}_0 + \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3, \\ \mathbf{f}_3, & \mathbf{f}_0 + \mathbf{f}_3, & \mathbf{f}_0 + \mathbf{f}_1 + \mathbf{f}_2, & 2\mathbf{f}_0 + \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3. \end{array}$$

σ -action. From the Dynkin diagram, the involution σ acts on \mathfrak{h} by $\mathbf{e}_4 \leftrightarrow -\mathbf{e}_4$, fixing the other \mathbf{e}_i . Thus $\mathfrak{h} = \mathbf{3}^+ \oplus \mathbf{1}^-$ where the superscript denotes the eigenvalue of each summand. Alternatively, σ interchanges $\mathbf{f}_2, \mathbf{f}_3$ and fixes $\mathbf{f}_0, \mathbf{f}_1$. From this it follows that $\mathfrak{g}_\pm = \mathbf{6}^+ \oplus (\mathbf{3}^+ \oplus \mathbf{3}^-)$, and hence

$$\mathfrak{so}(8) = \mathbf{21}^+ \oplus \mathbf{7}^- \quad \text{under the action of } \sigma.$$

1.4.1 *Remark.* Alternatively, one can see this from the fact that $\mathfrak{so}(8) \cong \bigwedge^2 V$ where σ acts on $V = \mathbb{C}^8$ by reflection in a hyperplane $H \subset V$. Then $\mathbf{21}^+ = \bigwedge^2 H$ and $\mathbf{7}^- = H$.

τ -action. τ cyclically permutes $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ and fixes \mathbf{f}_0 , so that $\mathfrak{h} = \mathbf{2} \oplus \mathbf{1}^\omega \oplus \mathbf{1}^{\omega^2}$ where $\omega = e^{2\pi i/3}$. For \mathfrak{g}_\pm , we observe that the entries of each of the first three columns of the array above are cyclically permuted by τ , while those of the last column are invariant. It follows that each of $\mathfrak{g}_\pm = \mathbf{6} \oplus \mathbf{3}^\omega \oplus \mathbf{3}^{\omega^2}$, and hence:

$$\mathfrak{so}(8) = \mathbf{14} \oplus \mathbf{7}^\omega \oplus \mathbf{7}^{\omega^2} \quad \text{under the action of } \tau.$$

1.4.2 *Remark.* The invariant 14-dimensional subspace is the Lie algebra of $G_2 = \text{Aut } \mathbb{O}$. This is well-known, and follows from (5).

2 Galois Spin(8) bundles

After some preliminary remarks on spin bundles in §2.1 we shall introduce the main objects of this paper, Galois Spin(8)-bundles, in §2.2 (see Definition 2.2.3).

2.1 Spin bundles

We next consider principal G -bundles $F \rightarrow X$ where X is a curve and $G = \text{Spin}(2n)$ an even complex spin group. Given a representation $\rho : G \rightarrow SL(W)$ we can form a vector bundle $W_F = F \times_\rho W$.

2.1.1 **LEMMA.** *If F_1 is a Spin(2m)-bundle and F_2 is a Spin(2n)-bundle, then there is a Spin(2m+2n)-bundle $F_1 + F_2$ with*

$$\begin{aligned} V_{F_1+F_2} &= V_{F_1} \oplus V_{F_2}, \\ S_{F_1+F_2}^+ &= S_{F_1}^+ \otimes S_{F_2}^+ \oplus S_{F_1}^- \otimes S_{F_2}^-, \\ S_{F_1+F_2}^- &= S_{F_1}^+ \otimes S_{F_2}^- \oplus S_{F_1}^- \otimes S_{F_2}^+. \end{aligned}$$

2.1.2 *Remark.* More generally, $\text{Nm}(F_1 + F_2) = \text{Nm}(F_1) \otimes \text{Nm}(F_2)$ if F_1 and F_2 are Clifford bundles and Nm is the spinor norm.

Proof. Since the spin groups are simply connected the inclusion $SO(2m) \times SO(2n) \hookrightarrow SO(2m + 2n)$, $(g, h) \mapsto g \oplus h$ has a unique lift λ making the following diagram commute:

$$\begin{array}{ccc} \text{Spin}(2m) \times \text{Spin}(2n) & \xrightarrow{\lambda} & \text{Spin}(2m + 2n) \\ 4 : 1 \downarrow & & \downarrow 2 : 1 \\ SO(2m) \times SO(2n) & \hookrightarrow & SO(2m + 2n). \end{array}$$

The bundle $F_1 + F_2$ is obtained by applying λ to transition functions of F_1 and F_2 with respect to a sufficiently fine open cover of the curve. The identification $V_{F_1+F_2} = V_{F_1} \oplus V_{F_2}$ is then immediate from the construction.

Denote by $V_1 = N_1 \oplus N_1^\vee$ and $V_2 = N_2 \oplus N_2^\vee$ the orthogonal representations of $\text{Spin}(2m)$ and $\text{Spin}(2n)$ respectively, decomposed into maximal isotropic subspaces. Then $N = N_1 \oplus N_2$ is maximal isotropic in $V = V_1 \oplus V_2$ and

$$\begin{aligned} \bigwedge^{\text{even}} N &= \bigwedge^{\text{even}} N_1 \otimes \bigwedge^{\text{even}} N_2 \oplus \bigwedge^{\text{odd}} N_1 \otimes \bigwedge^{\text{odd}} N_2, \\ \bigwedge^{\text{odd}} N &= \bigwedge^{\text{even}} N_1 \otimes \bigwedge^{\text{odd}} N_2 \oplus \bigwedge^{\text{odd}} N_1 \otimes \bigwedge^{\text{even}} N_2. \end{aligned}$$

From this it follows that the spinor bundles of $F_1 + F_2$ are as asserted. \square

If $P \subset G$ is a parabolic subgroup we can form the bundle of homogeneous spaces F/P with fibre G/P . The G -bundle F is said to be *stable* (resp. *semistable*) if for every maximal parabolic $P \subset G$ and every section $s : X \rightarrow F/P$ one has

$$\deg s^* T_{F/P}^{\text{vert}} > 0 \quad (\text{resp. } \geq 0)$$

where T^{vert} denotes the vertical tangent bundle. (See [17].) In the case of $G = \text{Spin}(2n)$ there are (up to conjugacy) n maximal parabolics and the spaces G/P are the grassmannians of isotropic subspaces $U \subset V$ of dimension $d = 1, \dots, n-2$ and the two spinor varieties of isotropic subspaces of dimension $d = n$. A section $s : X \rightarrow F/P$ corresponds to an isotropic subbundle $\underline{U} \subset V_F$ and it is easy to check that the stability condition reduces to the slope inequality

$$\mu(\underline{U}) < \mu(V_F) \quad \forall \text{ isotropic subbundles } \underline{U} \subset V_F.$$

In other words, F is a (semi)stable $\text{Spin}(2n)$ -bundle if and only if V_F is (semi)stable as an orthogonal vector bundle. (See also [14], Lemma 1.2.)

Recall that, just as for vector bundles, the moduli problem for spin bundles requires an equivalence relation on semistable bundles coarser than isomorphism, called S-equivalence. For the general definition in the context of principal bundles we refer to [9]; for our purposes the following remarks will be sufficient.

Suppose that W is an orthogonal vector bundle with a filtration

$$U_1 \subset U_2 \subset \dots \subset U_k \subset U_k^\perp \subset \dots \subset U_2^\perp \subset U_1^\perp \subset W,$$

where U_1, \dots, U_k are destabilising subbundles (i.e. isotropic of degree 0) and each ‘quotient’

$$(U_{i+1}/U_i) \oplus (U_i^\perp/U_{i+1}^\perp) \cong (U_{i+1}/U_i) \oplus (U_{i+1}/U_i)^\vee$$

is a stable orthogonal bundle, with its natural orthogonal structure (we include $U_0 = 0$). Clearly such a filtration always exists, and $k = 0$ if and only if W is stable. The graded orthogonal bundle

$$\text{gr}(W) = \bigoplus_{i=0}^{k-1} \{(U_{i+1}/U_i) \oplus (U_{i+1}/U_i)^\vee\} \oplus U_k^\perp/U_k$$

is independent of the filtration, and two orthogonal bundles W, W' of the same rank are said to be S-equivalent if and only if their graded bundles are isomorphic as orthogonal bundles:

$$W \sim W' \quad \Longleftrightarrow \quad \text{gr}(W) \cong \text{gr}(W').$$

Recall that by [16] Proposition 4.5 V_F is stable as an orthogonal bundle if and only if V_F is polystable as a vector bundle, i.e.

$$V_F = V_1 \oplus \dots \oplus V_k,$$

where the summands V_i are stable as vector bundles and nonisomorphic.

Let us now restrict to the case of $\text{Spin}(8)$, and suppose that the spinor bundles are polystable of the same shape, that is, with stable summands of the same ranks as the V_i :

$$\begin{aligned} S_F^+ &= S_1^+ \oplus \dots \oplus S_k^+, \\ S_F^- &= S_1^- \oplus \dots \oplus S_k^-. \end{aligned}$$

In this situation we shall need later on to understand the automorphism group of the spin bundle F . This is determined by its action on the three bundles above; by stability and orthogonality the automorphism group of each of these is μ_2^k where $\mu_2 = \{\pm 1\}$. $\text{Aut } F$ is then described by the following diagram with exact rows:

$$\begin{array}{ccccccc} 0 \rightarrow & \mu_2 & \xrightarrow{\beta} & \text{Aut } F & \xrightarrow{\alpha} & (\mu_2)^k & \rightarrow 0 \\ (10) & & \parallel & \uparrow \text{inclusion} & & \uparrow \text{diagonal} & \\ 0 \rightarrow & \mu_2 & \rightarrow & Z(\text{Spin}(8)) & \rightarrow & \mu_2 & \rightarrow 0. \end{array}$$

The map α is the representation on V_F , while $\beta(-1)$ acts as -1 on each of S_F^\pm and as $+1$ on V_F . Explicitly, the elements of $\text{Aut } F$ are the $4 \times 2^{k-1} = 2^{k+1}$ matrices (acting on $V_F \oplus S_F^+ \oplus S_F^-$)

$$(11) \quad \begin{pmatrix} \varepsilon & & \\ & \varepsilon & \\ & & \varepsilon \end{pmatrix}, \begin{pmatrix} \varepsilon & & \\ & -\varepsilon & \\ & & -\varepsilon \end{pmatrix}, \begin{pmatrix} -\varepsilon & & \\ & \varepsilon & \\ & & -\varepsilon \end{pmatrix}, \begin{pmatrix} -\varepsilon & & \\ & -\varepsilon & \\ & & \varepsilon \end{pmatrix},$$

where $\varepsilon \in (\mu_2)^k$ satisfies $\prod \varepsilon_i = 1$, i.e. contains an even number of -1 s (and in each matrix denotes a diagonal block).

2.2 Bundles with S_3 -action

Let X be a smooth curve acted on by the group S_3 . We shall assume that the action is faithful and that each involution has nonempty fixed-point set.

The group S_3 acts in two ways on isomorphism classes of principal $\text{Spin}(8)$ -bundles on X . First, it has a right ‘triality’ action $F \mapsto F^u$ (where $u \in S_3$) by outer automorphisms of the structure group. These are defined up to inner automorphisms, as in Lemma 1.1.3, but inner automorphisms preserve the isomorphism class of the bundle. Second, the right action of S_3 on X induces by pull-back a left action on bundles, $F \mapsto u(F) := (u^{-1})^*F$. We shall be interested in (semistable) bundles for which these two actions agree, i.e. the fixed-point set $\mathcal{F}_X \subset \mathcal{M}_X(\text{Spin}(8))$ of the S_3 -action $F \mapsto u^*F^u$, $u \in S_3$.

If $F \in \mathcal{F}_X$ then there is a nontrivial *Mumford group* \mathcal{G}_F consisting of pairs $g \in S_3$, $\lambda : g(F) \xrightarrow{\sim} F^g$ with multiplication law

$$(g, \lambda)(h, \mu) = (gh, \lambda^h \circ g\mu).$$

Here $\lambda^h : g(F^h) = g(F)^h \xrightarrow{\sim} F^{gh}$ and $g\mu : gh(F) \xrightarrow{\sim} g(F^h)$ are the natural induced isomorphisms. This group is an extension

$$(12) \quad 1 \longrightarrow \text{Aut } F \longrightarrow \mathcal{G}_F \longrightarrow S_3 \longrightarrow 1.$$

Note in particular that there is an S_3 -action on $\text{Aut } F$ induced, via conjugation in \mathcal{G}_F , by this sequence. It is easy to verify that the invariant subgroup $(\text{Aut } F)^{S_3}$ consists of $\alpha \in \text{Aut } F$ commuting with S_3 in the sense that for all $g \in S_3$ and $\lambda : g(F) \xrightarrow{\sim} F^g$ the following diagram commutes:

$$\begin{array}{ccc} gF & \xrightarrow{\lambda} & F^g \\ g\alpha \downarrow & & \downarrow \alpha^g \\ gF & \xrightarrow{\lambda} & F^g. \end{array}$$

2.2.1 Remark. If F is stable as a $\text{Spin}(8)$ bundle then the vector bundles V_F, S_F^\pm are polystable with k stable summands, say. The automorphism group $\text{Aut } F$ is then the group of order 2^{k+1} described in the last section ((10) and (11)). The action of S_3 on $\text{Aut } F$ determined by the sequence (12) permutes, for each $\varepsilon \in (\mu_2)^k$, $\prod \varepsilon_i = 1$, the last three matrices of (11) in the natural way. In particular we have

$$(13) \quad (\text{Aut } F)^{S_3} = \ker \{ \Pi : (\mu_2)^k \rightarrow \mu_2 \}.$$

2.2.2 LEMMA. Suppose $F \in \mathcal{F}_X$. Then the following data are equivalent:

1. A splitting $\Lambda : S_3 \rightarrow \mathcal{G}_F$ of the Mumford sequence (12).

2. A lift of the S_3 -action on X to the principal bundle F , in the sense that there is for each $u \in S_3$ a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\lambda(u)} & F^g \\ \downarrow & & \downarrow \\ X & \xrightarrow{g \in S_3} & X \end{array}$$

and that $\lambda(u)^w \circ \lambda(w) = \lambda(uw)$ for all $u, w \in S_3$.

3. An orthogonal lift of the S_3 -action on X to the Chevalley bundle $V_F \oplus S_F^+ \oplus S_F^-$ such that:
- (i) the 3-cycle $\tau \in S_3$ permutes the summands cyclically $V_F \xrightarrow{\sim} \tau^* S_F^+ \xrightarrow{\sim} (\tau^2)^* S_F^- \xrightarrow{\sim} V_F$;
 - (ii) the involution $\sigma \in S_3$ lifts to an involution $V_F \xrightarrow{\sim} \sigma^* V_F$, acting in the fibre $(V_F)_x$ at a fixed point $x \in X$ of σ as reflection in a hyperplane on which the quadratic form is nondegenerate, and exchanges the spinor bundles $S_F^+ \xrightarrow{\sim} \sigma^* S_F^-$.

Proof. The equivalence of 1 and 2 is essentially obvious and we leave its verification to the reader. To show that 2 is equivalent to 3 we shall represent the bundle F by a $\text{Spin}(8)$ -valued Čech cocycle $\{g_{ij}\}$ with respect to an open cover $\{U_i\}$ of the curve. We can assume that this open cover is invariant under the action of the finite group S_3 , and we shall denote the image of U_i under $w \in S_3$ by U_{iw} .

If $\{h_{ij}\}$ represents a second bundle F' then a bundle isomorphism $F \xrightarrow{\sim} F'$ is represented by a cochain $\{f_i\}$ satisfying

$$f_i g_{ij} = h_{ij} f_j.$$

For each $w \in S_3$ the bundle $w^* F$ has cocycle $\{w^*(g_{i^w j^w})\}$; while the bundle $F^{w^{-1}}$ has cocycle $\{w g_{ij} w^{-1}\}$, where the conjugation takes place in the semidirect product $\text{Spin}(8) \rtimes S_3 \subset O(V \oplus S^+ \oplus S^-)$. The condition that $F^{w^{-1}} \cong w^* F$ is therefore:

$$(14) \quad \exists \text{ Spin}(8)\text{-valued cochain } \{f_i\} \text{ satisfying } f_i^{-1} w^*(g_{i^w j^w}) f_j = w g_{ij} w^{-1}.$$

Next now suppose that the vector bundle $V_F \oplus S_F^+ \oplus S_F^-$ admits an orthogonal lift of the S_3 -action. It is easy to check that necessary and sufficient conditions for this are the existence, for each $w \in S_3$, of an $O(V \oplus S^+ \oplus S^-)$ -valued cochain $\{\tilde{f}_i\}$ satisfying

$$(15) \quad \tilde{f}_i g_{ij} = w^*(g_{i^w j^w}) \tilde{f}_j,$$

together with suitable compatibility assumptions on the cochains $\{\tilde{f}_i\}$ as $w \in S_3$ varies. We shall show that (15) reduces to (14) when the lift satisfies the conditions (i) and (ii) of the lemma. Namely, these conditions are equivalent to requiring that $\{\tilde{f}_i\}$ takes values in $\text{Spin}(8) \rtimes S_3 \subset O(V \oplus S^+ \oplus S^-)$ and projects to the element $w^{-1} \in S_3$. (Note that the 1-dimensional -1 -eigenspaces at the fixed points when w is an involution (3 part (ii) of the lemma) arise from the choices of reflections of $V \oplus S^+ \oplus S^-$ needed to define $\rho : S_3 \hookrightarrow O(V \oplus S^+ \oplus S^-)$.)

Consequently, for each $w \in S_3$ the cochain defining the lift of w^{-1} to the Chevalley bundle can be expressed as $\tilde{f}_i = (f_i, w^{-1})$ where f_i is $\text{Spin}(8)$ -valued. Then (using (3) in §1)

$$\begin{aligned}\tilde{f}_i g_{ij} &= (f_i, w^{-1})(g_{ij}, 1) \\ &= (f_i w g_{ij} w^{-1}, w^{-1})\end{aligned}$$

while

$$\begin{aligned}w^*(g_{i^w j^w}) \tilde{f}_j &= (w^*(g_{i^w j^w}), 1)(f_j, w^{-1}) \\ &= (w^*(g_{i^w j^w}) f_j, w^{-1}).\end{aligned}$$

So we see that (15) reduces to (14) as required. \square

2.2.3 DEFINITION. (i) By a *Galois $\text{Spin}(8)$ -bundle* we shall mean a pair (F, Λ) where F is a principal $\text{Spin}(8)$ -bundle on a curve X on which S_3 acts faithfully, and $\Lambda : S_3 \rightarrow \mathcal{G}_F$ is a lift of the group action to F in the sense of Lemma 2.2.2. Semistability and S -equivalence of Galois bundles will refer to the corresponding properties of the underlying bundles F .
(ii) Let \mathcal{N}_X be the set of S -equivalence classes of semistable Galois $\text{Spin}(8)$ -bundles; and let $\mathcal{F}_X \subset \mathcal{M}_X(\text{Spin}(8))$ denote the fixed point set of the S_3 -action $F \mapsto u^* F^u$, $u \in S_3$. For $F \in \mathcal{F}_X$ we denote by $\mathcal{N}(F)$ the fibre of the forgetful map

$$\mathcal{N}_X \rightarrow \mathcal{F}_X \subset \mathcal{M}_X(\text{Spin}(8)).$$

The set $\mathcal{N}(F)$ of Galois structures, if nonempty, is a torsor over the cohomology group $H^1(S_3, \text{Aut } F)$.

2.2.4 LEMMA. (S.M.J. WILSON) *Let G be a finite group and $H \subset G$ be a normal subgroup of coprime order and index. Suppose that G acts on an abelian group A such that H, A have coprime order. Then for all $q \geq 1$:*

$$\begin{aligned}H^q(H, A) &= 0, \\ H^q(G, A) &\cong H^q(G/H, A^H).\end{aligned}$$

Proof. As a \mathbb{Z} -module $H^q(H, A)$ is annihilated by $|A|$ trivially, and is also annihilated by $|H|$ since every element has order dividing $|H|$ by [10] p.117 Proposition 5.3. $H^q(H, A)$ is therefore annihilated by $\gcd(|H|, |A|) = 1$, proving the first part. It then follows from this and the Hochschild-Serre spectral sequence that for all $q \geq 1$ the inflation maps $H^q(G/H, A^H) \rightarrow H^q(G, A)$ are isomorphisms (see [10] p.355 Exercise 3). \square

In our situation $G = S_3$, $H = \langle \tau \rangle$, $A = \text{Aut } F$, the quotient G/H acts trivially on A^H , and it follows that

$$H^q(S_3, \text{Aut } F) \cong \ker \{ \Pi : (\mu_2)^k \rightarrow \mu_2 \} \quad \text{for } q = 1, 2.$$

From this we conclude:

2.2.5 PROPOSITION. *Suppose $F \in \mathcal{F}_X$ is stable with k summands (as a polystable vector bundle).*

- (i) *If $k = 1$ then there exists a unique Galois structure on F .*
- (ii) *If $k > 1$ then $H^2(S_3, \text{Aut } F) \cong (\mu_2)^{k-1}$. If the Mumford sequence (12) splits then the set $\mathcal{N}(F)$ of Galois structures on F has cardinality 2^{k-1} .*

2.2.6 *Remark.* If V_F , and hence also S_F^\pm , are stable as vector bundles (case (i) above) then one can see the uniqueness of the Galois structure directly. Consider isomorphisms $\alpha, \beta : V_F \xrightarrow{\sim} \sigma^* V_F$. By stability these differ by a scalar $\alpha = \lambda\beta$, $\lambda \in \mathbb{C}$, but at a fixed point both α, β act with 1-dimensional -1 -eigenspace in the fibre, and this forces $\lambda = 1$.

2.3 Further remarks

We should ask at this point how one can construct Galois bundles. Note that since F is determined by its spinor bundles S_F^\pm it is determined by the orthogonal bundle V_F :

$$S_F^+ = \tau(V_F), \quad S_F^- = \tau^2(V_F).$$

(Because of the relation $\sigma\tau = \tau^2\sigma$ (recall 1.1.1 for notation) these two bundles are necessarily interchanged by σ .) In the following lemma, which plays a key role in later sections, we construct a ‘split’ orthogonal bundle V_F which gives rise to a Galois spin bundle. We make use of the quotient

$$\pi : X \xrightarrow{2:1} Y = X/\sigma.$$

2.3.1 **LEMMA-DEFINITION.** *Let $E \rightarrow Y = X/\sigma$ be a rank 2 vector bundle with $\det E = \mathcal{O}_Y$. Then there is a unique Galois $\text{Spin}(8)$ -bundle $F = F_E \rightarrow X$ with*

$$V_F = \mathbb{C}^2 \otimes \pi^* E \oplus \tau(\pi^* E) \otimes \tau^2(\pi^* E).$$

Proof. Let $F_1 = \mathbb{C}^2 \otimes E'$ and $F_2 = \tau(E') \otimes \tau^2(E')$ for any rank 2 bundle $E' \rightarrow X$ with trivial determinant. Each of F_1, F_2 is a $\text{Spin}(4)$ -bundle whose spinor bundles are its two factors. So by Lemma 2.1.1, $F_1 \oplus F_2$ is the orthogonal representation of a $\text{Spin}(8)$ bundle F , and we have:

$$\begin{aligned} V_F &= F_1 \oplus F_2, \\ S_F^+ &= \mathbb{C}^2 \otimes \tau(E') \oplus \tau^2(E') \otimes E' = \tau(F_1 \oplus F_2), \\ S_F^- &= \mathbb{C}^2 \otimes \tau^2(E') \oplus E' \otimes \tau(E') = \tau^2(F_1 \oplus F_2). \end{aligned}$$

It follows that σ lifts to an isomorphism $S_F^+ \xrightarrow{\sim} \sigma^* S_F^-$. Moreover, σ lifts to an isomorphism $F_2 \xrightarrow{\sim} \sigma^* F_1$ having in the fibre at each fixed point $x \in X$ one-dimensional -1 -eigenspace $\bigwedge^2(\tau E')_x$. This extends to an isomorphism $V_F \xrightarrow{\sim} \sigma^* V_F$ with the same property provided E' is the pull-back of some bundle $E \rightarrow Y$; so we are done. Note that in this construction we must have $\det E = \mathcal{O}_Y$ since $\det E' = \mathcal{O}_X$: by hypothesis σ has nonempty fixed-point set, so π^* is injective on line bundles. \square

2.3.2 *Remark.* There are, of course, two more quotients of X by choosing $\sigma\tau$ or $\sigma\tau^2$ instead of σ . In fact these are both isomorphic to Y : the S_3 action on X induces an inclusion $X \hookrightarrow Y \times Y \times Y$, and the three quotients are then simply the projections on the three factors. Under the above construction, choosing a different projection has the effect of permuting the bundles V_F, S_F^\pm .

The existence of Galois bundles not of the form 2.3.1 (and stable, in fact—note that the bundle given by 2.3.1) is semistable but not stable: $\pi^*E \hookrightarrow V_F$ is a destabilising subbundle) will be seen in section 5.

Our next remark is a numerical observation that will reappear in later (see Remark 4.2.3(ii)).

We might relax the condition that S_3 acts faithfully on the curve X . Suppose, in particular, that the normal subgroup $\langle \tau \rangle \subset S_3$ acts trivially, i.e. that S_3 acts by a single involution of X . Then a Galois $\text{Spin}(8)$ -bundle F requires isomorphisms $V_F \xrightarrow{\sim} S_F^\pm$; let us interpret this as asking simply for a rank 8 orthogonal bundle $V_F \rightarrow X$ together with a lift of the involution $\sigma : X \leftrightarrow X$ which at fixed points acts in the fibre with 1-dimensional -1 -eigenspace.

Suppose in particular that $X \rightarrow \mathbb{P}^1$ is a hyperelliptic curve and σ the sheet-involution. Then provided X has genus $g \geq 4$ there is a projective moduli space \mathcal{M} of such $SO(8)$ -bundles which is described explicitly in [16] Theorem 1. This says that

$$\mathcal{M} \cong \text{Grass}_{g-4}(Q_1 \cap Q_2)/(\mathbb{Z}/2)^{2g+2},$$

where $Q_1, Q_2 \subset \mathbb{P}^{2g+1}$ are quadrics, and Grass_{g-4} denotes the grassmannian of projective $g-4$ -planes isotropic for both quadrics. In particular the dimension is easy to compute: \mathcal{M} is cut out in the grassmannian by a pair of sections of the bundle S^2U^\vee where U is the tautological bundle. Hence

$$\begin{aligned} \dim \mathcal{M} &= (g-3)(g+5) - (g-3)(g-2) \\ &= 7(g-3). \end{aligned}$$

Finally, we make some remarks concerning curves with $\mathbb{Z}/3$ -action—in particular, for example, cyclic trigonal curves. In this case we can certainly imitate the above constructions replacing S_3 by $\mathbb{Z}/3$, with triality action on $\text{Spin}(8)$ determined by making a choice (there are two) of embedding $\mathbb{Z}/3 \hookrightarrow S_3$. The conjugation action of S_3 on $\mathcal{M}_X(\text{Spin}(8))$ then restricts to $\mathbb{Z}/3$ and we can consider the fixed-point-set $\mathcal{F}_X^{\mathbb{Z}/3} \subset \mathcal{M}_X(\text{Spin}(8))$. A lifting of the $\mathbb{Z}/3$ -action is determined by a splitting of the ‘restricted’ Mumford group

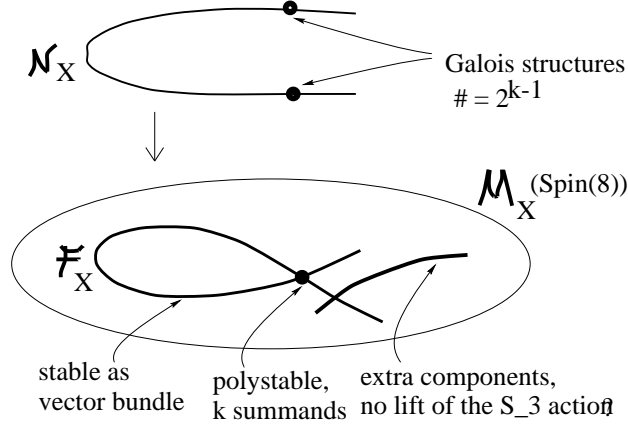
$$0 \rightarrow \text{Aut } F \rightarrow \mathcal{G}_F^{\mathbb{Z}/3} \rightarrow \mathbb{Z}/3 \rightarrow 0.$$

However, in this case the sequence *always splits uniquely* since by Lemma 2.2.4 $H^q(\mathbb{Z}/3, \text{Aut } F) = 0$ for $q = 1, 2$. So for our moduli space of $\mathbb{Z}/3$ -Galois bundles we can simply take $\mathcal{N}_X = \mathcal{F}_X$. Geometrically a Galois bundle is now a $\text{Spin}(8)$ -bundle F together with a lift of the (chosen triality) $\mathbb{Z}/3$ -action in the sense of 2.2.2(2), or equivalently 2.2.2(3) omitting the requirement (ii).

We expect that the results outlined in the introduction for the S_3 case should hold also for $\mathbb{Z}/3$. Indeed, the discussion of §3.1 in the next section goes through unimpeded, as does the dimension calculation of the moduli space in §4.3 (see Remark 4.3.2). However, the difficulty arises in computing the semistable boundary (§3.2), where we make essential use of the elements of order 2 in S_3 . Possibly one could get round this and prove a corresponding result for $\mathbb{Z}/3$, but we have not pursued the question here.

3 The moduli space

We wish to construct a moduli space \mathcal{N}_X of Galois $\text{Spin}(8)$ bundles on our S_3 -curve X . Using Proposition 2.2.5 this can be modelled on the fixed-point set $\mathcal{F}_X \subset \mathcal{M}_X(\text{Spin}(8))$ of the ‘conjugation’ action $F \mapsto u^{-1}F^u = u^*F^u$, $u \in S_3$. We shall show in this section that \mathcal{N}_X inherits from \mathcal{F}_X the structure of an analytic space which is smooth over stable spin bundles. \mathcal{N}_X and \mathcal{F}_X are locally isomorphic at stable vector bundles (i.e. at F such that V_F is stable as a vector bundle), while at polystable vector bundles \mathcal{N}_X resolves normal crossing singularities in \mathcal{F}_X :



We analyse the semistable boundary (i.e. Galois bundles nonstable as spin bundles) and show that, with the additional assumption that $X/S_3 \cong \mathbb{P}^1$, this consists precisely of equivalence classes of bundles of the form 2.3.1.

3.1 Local moduli

To begin, we examine the S_3 -action on the Kodaira-Spencer map at a stable spin bundle in $\mathcal{M}_X(\text{Spin}(8))$.

Suppose that F is a principal G -bundle F , for some reductive group G , represented by a Čech cocycle (transition functions) $\{g_{ij}\}$ with respect to an open cover $\{U_i\}$ of X . If $\text{ad } F$ is the vector bundle $F \times_{\text{ad}} \mathfrak{g}$ then a cohomology class $\xi \in H^1(X, \text{ad } F)$ is represented by a cocycle $\{\xi_{ij}\}$ of \mathfrak{g} -valued functions on $U_i \cap U_j$ satisfying

$$0 = \xi_{ij} + \xi_{ji}^{g_{ij}}, \quad 0 = \xi_{ij} + \xi_{jk}^{g_{kj}} + \xi_{ki}^{g_{ik}g_{kj}}, \quad \forall i, j, k;$$

where $\xi^g := \text{Ad}(g)\xi$ for $\xi \in \mathfrak{g}$ and $g \in G$. Exponentiating these conditions, i.e. applying $\exp : \mathfrak{g} \rightarrow G$, one gets precisely the cocycle conditions for transition functions $\{g_{ij} \exp(\xi_{ij})\}$. Denote the corresponding G -bundle by F_ξ . If the bundle F is stable then so is F_ξ in a neighbourhood of $0 \in H^1(X, \text{ad } F)$, and we have a rational map

$$\text{ks} : H^1(X, \text{ad } F) \rightarrow \mathcal{M}_X(G),$$

mapping $0 \mapsto F$ and complete in the sense of [17] Theorem 4.2. That is, ks induces an isomorphism from a neighbourhood of $0 \in H^1(X, \text{ad } F)/\Gamma_F$, where $\Gamma_F = \text{Aut } F/Z(G)$, to a neighbourhood of $F \in \mathcal{M}_X(G)$.

Now let $G = \text{Spin}(8)$, and suppose that (F, Λ) is a Galois bundle (Definition 2.2.3). Since the adjoint representation of $\text{Spin}(8)$ is preserved by triality it follows from Lemma 2.2.2(2) that Λ induces an orthogonal lift of the S_3 -action on X to the vector bundle $\text{ad } F$. (This is completely analogous to the lift of 2.2.2(3) to the Chevalley bundle; and indeed can be expressed in terms of 2.2.2(3), since as $\text{Spin}(8)$ -bundles $\text{ad } F = \bigwedge^2 V_F \cong \bigwedge^2 S_F^+ \cong \bigwedge^2 S_F^-$.) In particular Λ induces an action of S_3 on $H^1(X, \text{ad } F)$.

3.1.1 PROPOSITION. *Suppose that S_3 acts faithfully on the curve X and that (F, Λ) (where $\Lambda \in \mathcal{N}(F)$) is a stable Galois $\text{Spin}(8)$ -bundle. Then the Kodaira-Spencer map ks is S_3 -equivariant, where the S_3 -action on $H^1(X, \text{ad } F)$ is that induced by Λ and the action on $\mathcal{M}_X(\text{Spin}(8))$ is $E \mapsto u^* E^u$ for $u \in S_3$.*

The proof is a straightforward calculation with transition functions, using (14) and the cocycle conditions above for elements $\xi \in H^1(X, \text{ad } F)$, and we omit the details.

For each $(F, \Lambda) \in \mathcal{N}_X$ we shall denote the S_3 -invariant subspace of $H^1(X, \text{ad } F)$ by $H^1(X, \text{ad } F)^\Lambda$. It follows from the proposition that the Kodaira-Spencer map restricts to a map

$$\text{ks}_\Lambda : H^1(X, \text{ad } F)^\Lambda \rightarrow \mathcal{F}_X.$$

Suppose first that V_F is a stable vector bundle, so that (by Proposition 2.2.5) Λ is unique. The map ks_Λ factors to an injective map on the quotient $H^1(X, \text{ad } F)^\Lambda / \Gamma_F^\Lambda$ where $\Gamma_F^\Lambda = (\text{Aut } F)^{S_3} / Z(\text{Spin}(8))^{S_3}$. But from Remark 2.2.1 we know that $Z(\text{Spin}(8))^{S_3} = \{1\}$; and when V_F is a stable vector bundle $(\text{Aut } F)^{S_3}$ is also trivial. We therefore have a commutative diagram (more precisely, the horizontal maps are defined on neighbourhoods of 0):

$$\begin{array}{ccc} H^1(X, \text{ad } F) / \Gamma_F & \hookrightarrow & \mathcal{M}_X(\text{Spin}(8)) \\ \uparrow & & \uparrow \\ H^1(X, \text{ad } F)^\Lambda & \hookrightarrow & \mathcal{F}_X. \end{array}$$

The spaces \mathcal{N}_X and \mathcal{F}_X are therefore locally isomorphic, determining a smooth analytic structure on \mathcal{N}_X with tangent space $H^1(X, \text{ad } F)^\Lambda$.

3.1.2 Remark. In §5 we shall construct (for the case $X/S_3 \cong \mathbb{P}^1$) examples of $(F, \Lambda) \in \mathcal{N}_X$ for which the vector bundle V_F is stable.

More generally, for polystable V_F with k summands, consider the set of Galois structures $\mathcal{N}(F) = \{\Lambda_1, \dots, \Lambda_{2^{k-1}}\}$. This set is acted on transitively and faithfully by the group $(\text{Aut } F)^{S_3}$. The subgroup of Γ_F preserving each $H^1(X, \text{ad } F)^{\Lambda_i}$ is therefore trivial and so we have inclusions (from a neighbourhood of 0 to a neighbourhood of F):

$$H^1(X, \text{ad } F)^{\Lambda_i} \hookrightarrow \mathcal{F}_X; \quad i = 1, \dots, 2^{k-1}.$$

Each space can be viewed as Zariski tangent space of \mathcal{N}_X at (F, Λ_i) , and we conclude:

3.1.3 PROPOSITION. *The moduli space \mathcal{N}_X is a complex analytic space mapping holomorphically to $\mathcal{F}_X \subset \mathcal{M}_X(\text{Spin}(8))$. At a stable Galois bundle (F, Λ) it has Zariski tangent space*

$$T_{F, \Lambda} \mathcal{N}_X = H^1(X, \text{ad } F)^\Lambda.$$

We shall compute the dimension of the tangent space in section 4.3, and deduce (in the case of interest $X/S_3 \cong \mathbb{P}^1$) that \mathcal{N}_X is smooth at stable points (Theorem 4.2.2).

3.2 The semistable boundary

Our next task is to describe Galois bundles which are semistable but not stable. We shall assume from now on that the quotient X/S_3 is isomorphic to \mathbb{P}^1 , and under this assumption it turns out that up to S -equivalence these bundles are exactly those described in Lemma 2.3.1. To recall, let E be a rank 2 vector bundle on X pulled back from X/σ . We consider the Galois $\text{Spin}(8)$ -bundle $F_E = (V_E, S_E^+, S_E^-)$ where

$$(16) \quad \begin{aligned} V_E &= (\tau E)(\tau^2 E) \oplus E\mathbb{C}^2, \\ S_E^+ &= (\tau^2 E)E \oplus (\tau E)\mathbb{C}^2, \\ S_E^- &= E(\tau E) \oplus (\tau^2 E)\mathbb{C}^2. \end{aligned}$$

We have written $E\mathbb{C}^2 = E \otimes \mathbb{C}^2$ etc for brevity. Note that if a trivialisation of the determinant line bundle $\det E$ is fixed then fibrewise we are in the situation of §1.2: each rank 2 factor is naturally self-dual, taking adjugates is a well-defined involution of each rank 4 summand, and we have an explicit description of the Clifford multiplications among the three rank 8 bundles.

3.2.1 THEOREM. *Suppose that S_3 acts faithfully on a curve X with quotient $X/S_3 \cong \mathbb{P}^1$. A semistable Galois $\text{Spin}(8)$ bundle $F \rightarrow X$ is unstable if and only if it is of the form $F = F_E$ for some $E \in \mathcal{SU}_{X/\sigma}(2)$ up to S -equivalence.*

We have seen in §2.1 that stability of F is equivalent to stability of V_F as an orthogonal bundle; so suppose that $U \subset V_F$ is a destabilising subbundle. This means that U is an isotropic subbundle of degree 0. In particular it is semistable as a vector bundle and $\text{rank } U \leq 4$.

Using the lift of the S_3 -action on X to the Chevalley bundle $W_F = V_F \oplus S_F^+ \oplus S_F^-$ we can consider the image bundles $gU \subset W_F$ for group elements $g \in S_3$. In addition, we can consider the associated bundles (see §1.3)

$$\begin{aligned} A_U &= \text{Im } \{U \otimes S_F^+ \rightarrow S_F^-\}^\perp, \\ B_U &= \text{Im } \{U \otimes S_F^- \rightarrow S_F^+\}^\perp; \end{aligned}$$

as well as combinations of these constructions such as A_{A_U} , B_{gU} etc. We shall keep track of these subbundles of the Chevalley bundle using the results of §1.3.

To begin, suppose that $U \subset V_F$ has $\text{rank } U = 4$. Then one of $A_U \subset S_F^-$ or $B_U \subset S_F^+$ is a destabilising line subbundle, and so by the action of $\tau \in S_3$ we get a destabilising line subbundle in V_F .

If, on the other hand, $\text{rank } U = 3$ then it is—at least locally—the intersection of a pair of rank 4 isotropic subbundles parametrising isotropic subspaces of opposite families in the fibres.

Globally we obtain a rank 4 subbundle of the pull-back of V_F to an étale double cover of X ; however, the class of this double cover is the Stiefel-Whitney class $w_1(F)$, which is zero since F is spin. We therefore have a pair of rank 4 isotropic subbundles globally on X . Since U and V_F are semistable these rank 4 bundles have degree 0 and so again V_F has (two) destabilising line subbundles.

It is therefore sufficient to assume that $\text{rank } U \leq 2$. We shall consider for any subbundle $U \subset V_F$ the Clifford multiplication

$$\begin{array}{ccc} \tau\sigma U \otimes \tau^2 U & \hookrightarrow & S_F^+ \otimes S_F^- \\ \mu \searrow & & \downarrow \\ & & V_F \end{array}$$

3.2.2 LEMMA. *If $\text{rank } U = 1$ then $\mu = 0$; if $\text{rank } U = 2$ then $\text{rank } \mu \leq 1$.*

3.2.3 Remark. In what follows, we shall freely interpret the results of §1.3 as (Galois) bundle theoretic statements. The justification for this is that the Galois structure moves the fibres in just the right way for the global Clifford multiplications to make sense fibrewise. For example, if $U \subset V_F$ then consider the fibres U_x and $(\tau U)_x$ at a point $x \in X$. By definition

$$(\tau U)_x = U_{\tau^{-1}(x)} \subset (V_F)_{\tau^{-1}(x)} = (S_F^+)_{\tau^{-1}(x)},$$

so that there is a well-defined multiplication map $U_x \otimes (\tau U)_x \rightarrow (S_F^-)_x$.

Proof of 3.2.2. Apply Lemma 1.1.5 to $\sigma U \subset V_F$ with $g = \tau\sigma$. This says that the composition $\sigma U \otimes \tau U \hookrightarrow V_F \otimes S_F^+ \rightarrow S_F^-$ has rank 0 if U is a line bundle and rank ≤ 1 if U has rank 2. Now apply τ and use the S_3 equivariance of the Chevalley algebra structure. \square

We consider first a destabilising subbundle $E \subset V_F$ of rank 2. Observe that if μ has rank 1 then the image of μ is a destabilising line subbundle; we postpone this case until after Proposition 3.2.4 and assume that $\mu = 0$. (The fact that the image of μ is isotropic can be seen from the description of the Clifford multiplication map given in section 1.2.)

The second thing to observe is that, with respect to the quadratic form,

$$E^\perp/E \cong A_E \otimes B_E.$$

This is because E^\perp/E is an orthogonal bundle of rank 4 whose isotropic 2-planes in the generic fibre are parametrised by A_E, B_E respectively. (See §1.3).

3.2.4 PROPOSITION. *If V_F has no destabilising line subbundle and $E \subset V_F$ is a destabilising rank 2 subbundle then F is S -equivalent to F_E (16), where E is the pull-back of a stable rank 2 vector bundle on X/σ .*

Proof. We already have the S-equivalence $V_F \sim E^\vee \oplus E \oplus A_E \otimes B_E$; we examine the consequences of the vanishing of μ . Namely, $\tau^2 E = B_{\tau\sigma E}$ by Lemma 1.3.2, and so $E = B_{\tau^2\sigma E}$ by Lemma 1.3.3. So by Lemma 1.3.1 we have

$$A_E = A_{B_{\tau^2\sigma E}} = \tau^2 \sigma E.$$

Similarly $\sigma E = \sigma B_{\tau^2\sigma E} = A_{\sigma\tau^2\sigma E} = A_{\tau E}$ and so

$$B_{\sigma E} = B_{A_{\tau E}} = \tau E.$$

Next, $E^\vee \cong \Delta^{-1} \otimes E$ where $\Delta = \det E$, and so we have

$$\begin{aligned} V_F &\sim (\mathcal{O} \oplus \Delta^{-1})E \oplus A_E B_E, \\ S_F^+ &\sim (\mathcal{O} \oplus \Delta^{-1})A_E \oplus B_E E, \\ S_F^- &\sim (\mathcal{O} \oplus \Delta^{-1})B_E \oplus E A_E. \end{aligned}$$

We now impose the Galois condition. $S_F^+ = \tau V_F$ implies that Δ is τ -invariant ($\mathcal{O} \oplus \Delta^{-1}$ is the only split rank 2 factor since E, A_E, B_E are all stable). Similarly Δ is σ -invariant since $V_F = \sigma V_F$; it therefore descends to a line bundle on \mathbb{P}^1 , and since it has degree 0 this forces $\Delta = \mathcal{O}_X$.

Similarly $\sigma E \cong E$, and it follows from the above $A_E = \tau^2 E$, $B_E = \tau E$. As before, we note that since the -1 -eigenspaces at fixed points of σ lie in the fibres of $A_E B_E$, the involution must act trivially in the fibres of E , which therefore descends. \square

The case of a destabilising line subbundle, to which we turn now, involves a bit more work.

3.2.5 PROPOSITION. *Suppose that a line subbundle $L \subset V_F$ is isotropic of degree 0, and that the Clifford multiplication $\tau L \otimes \tau^2 L \rightarrow V_F$ vanishes. Then $F \sim F_E$ for $E = L \oplus L^{-1}$.*

Proof. By (7) in §1.3 and the hypothesis of the proposition, we have an isotropic subbundle $U = A_{\tau L} \cap B_{\tau^2 L} \subset V_F$ of rank 3. Using Lemma 1.3.3, moreover, one sees that $L \subset U$. Accordingly we have an S-equivalence

$$V_F \sim L \oplus L^\vee \oplus U/L \oplus (U/L)^\vee \oplus U^\perp/U.$$

We shall identify the last term first; note that $U^\perp = A_{\tau L} + B_{\tau^2 L}$, so we have to compute the rank 1 quotients $A_{\tau L}/U$ and $B_{\tau^2 L}/U$.

From (8) it follows that there is a short exact sequence of vector bundles

$$0 \rightarrow \tau L \otimes A_{\tau L} \rightarrow \tau L \otimes V_F \rightarrow B_{\tau L} \rightarrow 0.$$

Twisting by τL^{-1} and using the fact that $\tau L \otimes \tau^2 L \rightarrow V_F$ is zero we get short exact sequences (in which the vertical arrows are inclusions):

$$\begin{array}{ccccccc} 0 \rightarrow & A_{\tau L} & \rightarrow & V_F & \rightarrow & \tau L^{-1} \otimes B_{\tau L} & \rightarrow 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \rightarrow & U & \rightarrow & B_{\tau^2 L} & \rightarrow & \tau L^{-1} \otimes \tau^2 L & \rightarrow 0, \end{array}$$

and similarly

$$\begin{array}{ccccccc} 0 & \rightarrow & B_{\tau^2 L} & \rightarrow & V_F & \rightarrow & \tau^2 L^{-1} \otimes A_{\tau^2 L} \rightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & U & \rightarrow & A_{\tau L} & \rightarrow & \tau^2 L^{-1} \otimes \tau L \rightarrow 0. \end{array}$$

We deduce that

$$V_F \sim (L \oplus \tau^2 L^{-1} \otimes \tau L \oplus U/L) \oplus (\text{same bundle})^\vee.$$

To impose the condition that F is a Galois bundle we shall compute directly the spinor bundles from this expression and (1) in §1.1. First note that $\det A_{\tau L} = \det B_{\tau L} = \tau L^2$, so from the exact sequences above $\det U = \tau L \otimes \tau^2 L$ and hence

$$\det(U/L) = L^{-1} \tau L \tau^2 L.$$

Writing $N = L \oplus \tau L \tau^2 L^{-1} \oplus U/L$, the spinor bundles are, up to a line bundle, $S_F^+ = \bigwedge^{\text{even}} N$ and $S_F^- = \bigwedge^{\text{odd}} N$. We find that to get trivial determinant one must twist by τL^{-1} , and the result is then (writing $E = L \otimes (U/L)$ and using the fact that (since it has rank 2) $(U/L)^\vee = \tau L^{-1} \tau^2 L^{-1} \otimes E$):

$$\begin{aligned} V_F &\sim L \oplus L^{-1} \oplus \tau L \tau^2 L^{-1} \oplus \tau L^{-1} \tau^2 L \oplus U/L \oplus (U/L)^\vee \\ &= L \oplus L^{-1} \oplus \tau L \tau^2 L^{-1} \oplus \tau L^{-1} \tau^2 L \oplus (L^{-1} \otimes E) \oplus (\tau L^{-1} \tau^2 L^{-1} \otimes E), \\ S_F^+ &\sim \bigwedge^{\text{even}} N \\ &= \tau L \oplus \tau L^{-1} \oplus \tau^2(L) L^{-1} \oplus \tau^2(L^{-1}) L \oplus (\tau L^{-1} \otimes E) \oplus (L^{-1} \tau^2 L^{-1} \otimes E), \\ S_F^- &\sim \bigwedge^{\text{odd}} N \\ &= \tau^2 L \oplus \tau^2 L^{-1} \oplus L \tau L^{-1} \oplus L^{-1} \tau L \oplus (\tau^2 L^{-1} \otimes E) \oplus (L^{-1} \tau L^{-1} \otimes E). \end{aligned}$$

Since F is a Galois bundle these three vector bundles are permuted cyclically by τ , and it follows that the bundle $E = L \otimes U/L$ is τ -invariant. On the other hand, E is also σ -invariant (since U is), and σ acts as the identity in the fibres of E at fixed points since the -1 -eigenspace in the fibre of V_F is 1-dimensional and nonisotropic. It follows that E descends to a bundle on \mathbb{P}^1 (in fact the trivial bundle, by semistability of V_F , and can be written (notice that both summands are trivial)

$$E = \mathcal{O} \oplus L \tau L \tau^2 L.$$

It follows directly that

$$V_F \sim (L \oplus L^{-1} \oplus \tau L \tau^2 L \oplus \tau L \tau^2 L^{-1}) \oplus (\text{same bundle})^\vee$$

and that F has the desired form. □

We next show that if V_F has a destabilising line subbundle then this can be chosen with the property of Proposition 3.2.5.

3.2.6 LEMMA. *Suppose that a destabilising line subbundle $L \subset V_F$ is polar, with respect to the quadratic form on V_F , to its image $\sigma L \subset V_F$. Then the Clifford multiplication $\tau L \otimes \tau^2 L \rightarrow V_F$ vanishes.*

Proof. If $\sigma L = L$ then the result follows from Lemma 3.2.2, so we may assume that $L, \sigma L$ are distinct subbundles. In this case the bundle map $L \oplus \sigma L \rightarrow V_F$ is injective by semistability (the image is isotropic since we assume the summands are polar). In other words L and σL are distinct in all fibres.

If the multiplication $\tau L \otimes \tau^2 L \rightarrow V_F$ is nonzero then it must be injective (again by semistability, since the image is isotropic). This implies that $A_{\tau L} \cap B_{\tau^2 L} \subset V_F$ has rank 1 (using (7)). On the other hand it contains σL (by Lemma 3.2.2), and hence

$$\sigma L = A_{\tau L} \cap B_{\tau^2 L}.$$

We now consider the isotropic subbundles

$$U = A_{\tau\sigma L} \cap B_{\tau^2 L}, \quad E = A_{\tau\sigma L} \cap A_{\tau L}.$$

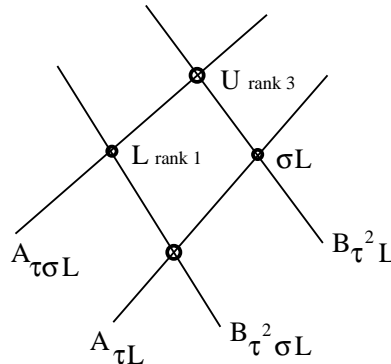
By Lemma 3.2.2 U has rank 3. By §1.3 (9), polarity of L and σL we see also that $E = \tau(A_{\sigma L} \cap A_L)$ has rank 2. But E, U are both contained in the rank 4 bundle $A_{\tau\sigma L}$, so

$$N = U \cap E = A_{\tau\sigma L} \cap A_{\tau L} \cap B_{\tau^2 L}$$

is an isotropic line subbundle. (E is not contained in U since then $A_{\tau L} \cap B_{\tau^2 L}$ would be ≥ 2 .) But $N \subset A_{\tau L} \cap B_{\tau^2 L} = \sigma L$, so in particular $\sigma L \subset A_{\tau\sigma L}$. Applying σ (and using Lemma 1.3.3) this shows that $L \subset B_{\tau^2 L}$. Applying τ^2 then shows that $\tau^2 L \subset B_{\tau L}$, i.e. that $\tau L \otimes \tau^2 L \rightarrow V_F$ is the zero map, a contradiction. \square

3.2.7 PROPOSITION. *If V_F admits a destabilising line subbundle then it has such a line subbundle $L \subset V_F$ for which $\tau L \otimes \tau^2 L \rightarrow V_F$ vanishes.*

Proof. We assume that $L \subset V_F$ is a destabilising line subbundle without this property, i.e. for which $\tau L \otimes \tau^2 L \rightarrow V_F$ is injective; and by the previous lemma we can assume that L is nowhere polar to σL , i.e. that the homomorphism $L \otimes \sigma L \rightarrow \mathcal{O}$ induced by the quadratic form on V_F is an isomorphism. By §1.3 (9) it follows that in this case $V_F = A_{\tau\sigma L} \oplus A_{\tau L}$. We let $U = A_{\tau\sigma L} \cap B_{\tau^2 L}$, and we have a configuration of subbundles:



Note in particular that

$$A_{\tau L} \cap B_{\tau^2 L} = \tau L \tau^2 L = \sigma L \cong L^{-1}.$$

As in the proof of Proposition 3.2.5 we have exact sequences (the vertical arrows are injective)

$$\begin{array}{ccccccc} 0 & \rightarrow & A_{\tau\sigma L} & \rightarrow & V_F & \rightarrow & \tau\sigma L^{-1} \otimes B_{\tau\sigma L} \rightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & U & \rightarrow & B_{\tau^2 L} & \rightarrow & \tau\sigma L^{-1} \otimes \tau^2 L \rightarrow 0. \end{array}$$

From this we see that $\det U = \tau^2 L \tau L^{-1}$ and

$$\bigwedge^2 U = U^\vee \otimes \det U = U^\vee \otimes \tau^2 L \tau L^{-1}.$$

Using these facts and §1.1 (1) we deduce that

$$\begin{aligned} V_F &= (L \oplus U) \oplus (L \oplus U)^\vee \\ S_F^+ &= (\tau L \oplus L \tau L \otimes U) \oplus (\tau L \oplus L \tau L \otimes U)^\vee \\ S_F^- &= (\tau^2 L \oplus \tau L \otimes U) \oplus (\tau^2 L \oplus \tau L \otimes U)^\vee. \end{aligned}$$

We now impose the condition that F is a Galois bundle to deduce that

$$\tau U = L \tau L \otimes U.$$

Also $\sigma U = U$ since σ interchanges $A_{\tau\sigma L}$ and $B_{\tau^2 L}$ (by Lemma 1.3.3). At fixed points σ acts trivially in the fibres of U since the latter is isotropic, and hence the projective bundle $\mathbb{P}(U)$ is S_3 -invariant and descends to a \mathbb{P}^2 -bundle on \mathbb{P}^1 . This is necessarily the projectivisation of a split vector bundle, and hence $U = P \oplus Q \oplus R$ for some line bundles P, Q, R on X . Note that these line bundles need not themselves descend to \mathbb{P}^1 may be permuted and twisted by the action of S_3 . However, since U is σ -invariant at least one summand must also be σ -invariant, and this summand will be a destabilising subbundle of V_F with the desired properties. \square

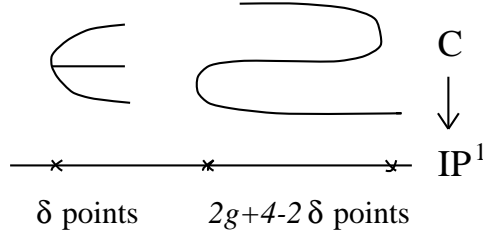
Proof of Theorem 3.2.1. By Proposition 3.2.4 we are reduced to the case where V_F has a destabilising line subbundle $L \subset V_F$. By Proposition 3.2.7 we can assume that Clifford multiplication on $\tau L \otimes \tau^2 L$ is zero, and by 3.2.5 this implies that $V_F \sim \mathbb{C}^2 E \oplus \tau E \tau^2 E$ with $E = L \oplus L^{-1}$. We just have to check that E descends to a bundle on X/σ . On the other hand the Galois condition tells us that $V_F \sim \mathbb{C}^2 \sigma E \oplus \tau^2 \sigma E \tau \sigma E$, and this implies that $\sigma E \cong E$. At a fixed point $x \in X$ of σ the 1-dimensional -1 -eigenspace of σ in the fibre of V_F is $\bigwedge^2(\tau E_x)$ (since τE and $\tau^2 E$ are interchanged by σ), and so in particular σ acts trivially in the fibre E_x . This shows that E is pulled back from a bundle on X/σ , as required. \square

4 Triality and trigonality

In this section (in §4.2) we shall introduce the main object of the paper, which is the moduli space \mathcal{N}_C of Galois $\text{Spin}(8)$ -bundles on the Galois closure of a trigonal curve $C \xrightarrow{3:1} \mathbb{P}^1$. Unless the curve is cyclic this Galois curve is a connected double cover of C , and we are interested in the Galois bundles on this double cover. The moduli space \mathcal{N}_C is an extension of the variety $\mathcal{S}U_C(2)$, which contains its singular locus.

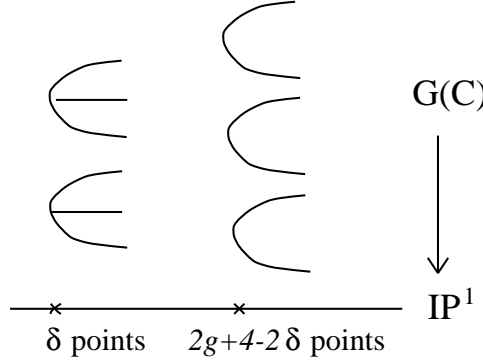
4.1 Galois trigonal curves

Let C be a nonhyperelliptic curve of genus $g \geq 3$ with a fixed trigonal pencil g_3^1 . We shall denote by $\mathcal{B} \subset \mathbb{P}^1$ the branch divisor; by Riemann-Hurwitz this has degree $2g + 4$. We shall suppose that \mathcal{B} contains δ points of multiplicity two, i.e. at which the cover has cyclic triple branching. There are then $2g + 4 - 2\delta$ simple branch points:



We shall need to distinguish the case of a cyclic cover $C \xrightarrow{3:1} \mathbb{P}^1$, in which case $\delta = g + 2$. (We shall see in a moment that this is also a sufficient condition.)

We construct a Galois cover $\mathcal{G}(C) \rightarrow \mathbb{P}^1$ as follows. $\mathcal{G}(C) \subset C \times C$ is the closure of the set $\{(q, r) \mid q \neq r, p + q + r \in g_3^1 \text{ for some } p \in C\}$. This is a double cover $\mathcal{G}(C) \rightarrow C$ by $(q, r) \mapsto p = g_3^1 - q - r$, and is smooth with branching behaviour:



Notice that over each divisor $p + q + r \in g_3^1$ the points of $\mathcal{G}(C)$ correspond to the orderings of $p, q, r \in C$. Thus the permutation group S_3 acts on $\mathcal{G}(C)$ on the right; in particular C is the quotient of $\mathcal{G}(C)$ by the 2-cycle $\sigma = (23)$. In general the general case S_3 is the Galois group of $\mathcal{G}(C)$; however, in the cyclic case $\mathcal{G}(C)$ breaks up into two connected components, each a copy of C acted on by $\mathbb{Z}/3$.

Restricting for a moment to the Galois S_3 case, we find by Riemann-Hurwitz that $\mathcal{G}(C)$ has genus $3g + 1 - \delta$. Over simple branch points in $\mathcal{B} \subset \mathbb{P}^1$ the three ramification points are respective fixed points of the three 2-cycles in S_3 . Over each double branch point there are two fixed points of the cyclic subgroup $\langle \tau \rangle \cong \mathbb{Z}/3$, at which τ acts in the tangent space as ω, ω^2 respectively, where $\omega = e^{2\pi i/3}$ is a primitive cube root of unity.

Consider now the quotient $H' = \mathcal{G}(C)/\langle \tau \rangle$. This is a hyperelliptic curve branched over $\mathcal{B} \subset \mathbb{P}^1$ and singular over the δ double points of \mathcal{B} . By Riemann-Hurwitz (again, we are assuming $\mathcal{G}(C)$ is a connected Galois S_3 curve) its normalisation H has Euler characteristic $e(H) = 2\delta - 2g$. If $\delta = g + 2$ then $e(H) = 4$ so H is a union of two lines and $\mathcal{G}(C)$ is a pair of copies of C , a contradiction. We have therefore shown:

4.1.1 LEMMA. C is cyclic over \mathbb{P}^1 if and only if $\delta = g + 2$.

If $\delta \leq g + 1$, on the other hand, then $g(H) = g + 1 - \delta$. We can summarise the situation in the following diagram.

$$(17) \quad \begin{array}{ccccc} & & 3g+1-\delta \mathcal{G}(C) = X & & \\ & \swarrow & & \searrow \pi & \\ g+1-\delta H' & & & & {}^g C = Y \\ & \searrow & & \swarrow g_3^1 & \\ & & \mathbb{P}^1 & & \end{array}$$

In the rest of the paper we shall apply the discussion of the previous sections to $X = \mathcal{G}(C)$ and $Y = C$, in the general trigonal case, and to $X = C$ when this is a cyclic trigonal curve. Whenever we refer to $\mathcal{G}(C)$ we shall always assume that we are in the general case.

4.2 Trigonal Galois bundles

We now turn to the moduli space we are interested in. We assume that C is a non-cyclic trigonal curve and we let $\mathcal{N}_C := \mathcal{N}_{\mathcal{G}(C)} \subset \mathcal{M}_{\mathcal{G}(C)}(\text{Spin}(8))$. (Though see Remark 4.2.3(i) regarding the cyclic case.)

4.2.1 *Remark.* If F is such a Galois bundle then the sheet-involution of $\mathcal{G}(C)$ over C , corresponding to $\sigma \in S_3$, lifts to the orthogonal bundle V_F ; applying the 3-cycle $\tau \in S_3$ one sees that the fibres of V_F over any $p + q + r \in g_3^1$ can be identified (though not canonically) with the three representations V, S^+, S^- . If F is semistable but not stable then $F = F_E$ for some $E \in \mathcal{SU}_C(2)$ (by Theorem 3.2.1). The following table shows explicitly how, in this case, the fibres of V_F, S_F^\pm are constructed from those of the rank 2 bundle E over a divisor $p + q + r \in g_3^1$. We abbreviate $\mathbb{C}^2 \otimes E_p \oplus E_q \otimes E_r$ to $2E_p + E_q E_r$:

C	$\mathcal{G}(C)$	V_F	S_F^+	S_F^-
p	(q, r)	$2E_p + E_q E_r$	$2E_q + E_r E_p$	$2E_r + E_p E_q$
	(r, q)	$2E_p + E_r E_q$	$2E_r + E_q E_p$	$2E_q + E_p E_r$

The following is the main result of this paper:

4.2.2 THEOREM. For any non-cyclic trigonal curve C the moduli space \mathcal{N}_C is smooth of dimension $7g - 14$ away from its semistable boundary $\mathcal{SU}_C(2)$.

4.2.3 *Remarks.* (i) We have already remarked in §2.3 that most of our constructions work also for cyclic trigonal curves. to be precise, if C is cyclic we define $\mathcal{N}_C = \mathcal{F}_C \subset \mathcal{M}_C(\text{Spin}(8))$, the fixed-point-set of the conjugation action of $\mathbb{Z}/3$ after choosing an embedding $\mathbb{Z}/3 \hookrightarrow S_3$. In this case both the smoothness (at stable points) and the dimension statements in Theorem 4.2.2 remain valid (see Remark 4.3.2 below); what is missing is the identification of the semistable boundary with $\mathcal{SU}_C(2)$ (though it is tempting to expect that this also remains true).

- (ii) Notice that the moduli space of Galois bundles on the hyperelliptic curve H of diagram (17) has dimension $7g - 14 - 7\delta$ (see §2.3); when $\delta = 0$ and $H' = H$ is smooth this number is $7g - 14$. This raises the question of the relationship between these two moduli spaces. Are they isomorphic, for example, or birational?

In view of Proposition 3.1.3, Theorem 3.2.1 and the fact that in the trigonal situation the quotient X/σ is just the curve C itself, all that remains to prove of Theorem 4.2.2 is the dimension statement; this will be carried out next.

4.3 Dimension calculation

By Proposition 3.1.3 we have to compute the dimension of the invariant subspace $H^1(\mathcal{G}(C), \text{ad } F)^{S_3}$ (or in the cyclic case $H^1(C, \text{ad } F)^{\mathbb{Z}/3}$). We shall do a little more, in fact, and compute the decomposition of $H^1(\mathcal{G}(C), \text{ad } F)$ into irreducible representations under this group action. In the same way, of course, we could as easily compute $T_F \mathcal{N}_X$ at a stable Galois bundle for any X , but we shall stick to the trigonal situation.

We shall consider the S_3 case first and return to $\mathbb{Z}/3$ in a moment. For use here and below we recall the following result of [1] Theorem 4.12.

4.3.1 ATIYAH-BOTT FIXED POINT THEOREM. *Suppose that $\gamma : X \rightarrow X$ is an automorphism of a compact complex manifold X with a finite set $\text{Fix}(\gamma) \subset X$ of fixed points, and suppose that γ lifts to a holomorphic vector bundle $E \rightarrow X$. Then*

$$\sum (-1)^p \text{trace } \gamma|_{H^p(X, E)} = \sum_{x \in \text{Fix}(\gamma)} \frac{\text{trace } \gamma|_{E_x}}{\det(1 - d\gamma_x)}.$$

Suppose now that $F \in \mathcal{N}_C$ is a stable Galois $\text{Spin}(8)$ -bundle for a general trigonal curve C . We use Atiyah-Bott to compute the trace of the group elements $\sigma, \tau \in S_3$ acting on $H^1(\mathcal{G}(C), \text{ad } F)$. By stability $H^0(\mathcal{G}(C), \text{ad } F) = 0$ and so for nontrivial $\gamma \in S_3$:

$$\text{trace } \gamma|_{H^1(\mathcal{G}(C), \text{ad } F)} = - \sum_{x \in \text{Fix}(\gamma)} \frac{\text{trace } \gamma|_{\text{ad } F_x}}{\det(1 - d\gamma_x)}.$$

$\gamma = \sigma$: σ has $2g + 4 - 2\delta$ fixed points, and at each of these $d\sigma_x = -1$. On $\text{ad } F_x$ we need to compute trace σ . But the bundle F is Galois, so this is just the trace of σ in the triality action on \mathfrak{so}_8 , which by section 1.4 is 14. Thus

$$\text{trace } \sigma|_{H^1(\mathcal{G}(C), \text{ad } F)} = -7(2g + 4 - 2\delta).$$

$\gamma = \tau$: we have seen that τ has δ *pairs* of fixed points, at which in the tangent space it acts as ω and ω^2 respectively. From section 1.4, on the other hand, trace $\tau = 7$ in the adjoint representation. So by Atiyah-Bott (and using $1 + \omega + \omega^2 = 0$) we get

$$\text{trace } \tau|_{H^1(\mathcal{G}(C), \text{ad } F)} = -7\delta \left\{ \frac{1}{1 - \omega} + \frac{1}{1 - \omega^2} \right\} = -7\delta.$$

Note that $\dim H^1(\mathcal{G}(C), \text{ad } F) = 28(3g - \delta)$ (since $\mathcal{G}(C)$ has genus $3g + 1 - \delta$); and so we have computed the character of $H^1(\mathcal{G}(C), \text{ad } F)$ under the S_3 action.

Recall that S_3 has three irreducible representations \mathbb{C} , \mathbb{C}_ε , \mathbb{C}_ρ^2 and character table:

$$(18) \quad \begin{array}{c|ccc} & 1 & \sigma & \tau \\ \hline \mathbf{1} & 1 & 1 & 1 \\ \varepsilon & 1 & -1 & 1 \\ \rho & 2 & 0 & -1 \end{array}$$

Here $\mathbb{C} = \mathbb{C}_1$ is the trivial representation, \mathbb{C}_ε the sign representation and \mathbb{C}_ρ^2 the 2-dimensional reflection representation.

Denote the character of $H^1(\mathcal{G}(C), \text{ad } F)$ by $a\mathbf{1} + b\varepsilon + c\rho$. Comparing with the character table gives:

$$\begin{aligned} a + b + 2c &= 28(3g - \delta) \\ a - b &= -7(2g + 4 - 2\delta) \\ a + b - c &= -7\delta. \end{aligned}$$

Solving these equations yields $a = 7g - 14$, $b = 21g + 14 - 14\delta$, $c = 28g - 7\delta$, and hence an S_3 -decomposition:

$$H^1(\mathcal{G}(C), \text{ad } F) = \mathbb{C}^{7g-14} \oplus (21g + 14 - 14\delta)\mathbb{C}_\varepsilon \oplus (28g - 7\delta)\mathbb{C}_\rho^2.$$

4.3.2 Remark. One can make an analogous calculation in the case of a cyclic trigonal curve. Here we apply the same method as above, with group $\langle \tau \rangle \cong \mathbb{Z}/3$ acting on C itself. Pick a stable Galois $\text{Spin}(8)$ -bundle $F \in \mathcal{N}_C$. As above we compute the trace of the group elements τ, τ^2 acting on $H^1(C, \text{ad } F)$, where by stability $H^0(C, \text{ad } F) = 0$.

By Atiyah-Bott:

$$\begin{aligned} \text{trace } \tau|_{H^1(C, \text{ad } F)} &= -7(g + 2)/(1 - \omega), \\ \text{trace } \tau^2|_{H^1(C, \text{ad } F)} &= -7(g + 2)/(1 - \omega^2). \end{aligned}$$

Suppose that the eigenspace decomposition under τ is $H^1(C, \text{ad } F) = \mathbb{C}^a \oplus \mathbb{C}^b \oplus \mathbb{C}^c$ for eigenvalues $1, \omega, \omega^2$ respectively. Then we have to solve the equations

$$\begin{aligned} a + b + c &= 28(g - 1) \\ a + \omega b + \omega^2 c &= -7(g + 2)/(1 - \omega) \\ a + \omega^2 b + \omega c &= -7(g + 2)/(1 - \omega^2). \end{aligned}$$

Adding the three equations gives again $a = 7g - 14$.

5 A nonabelian Schottky configuration

We shall show in this section that the moduli space \mathcal{N}_C contains a nonabelian ‘fattening’ of the classical Schottky configuration for trigonal curves. The Schottky configuration is the union of Prym Kummer varieties embedded in $\mathcal{SU}_C(2)$; when C is trigonal two things happen: (a) each Prym is, by the Recillas correspondence, the Jacobian of a tetragonal curve R_η , and (b) $\mathcal{SU}_C(2)$ is by Theorem 4.2.2 the ‘singular locus’ of a bigger moduli space \mathcal{N}_C . We show that \mathcal{N}_C contains a configuration of the moduli varieties $\mathcal{SU}_{R_\eta}(2)$ —each singular, of course, along the corresponding Prym Kummer.

5.1 The Schottky configuration

For each nonzero 2-torsion point $\eta \in J_C[2] \setminus \{\mathcal{O}\}$ we have an associated unramified double cover

$$f : C_\eta \rightarrow C.$$

We shall denote by φ the involution of C_η given by sheet-interchange over C ; it will denote also the induced involution of $\text{Pic}(C_\eta)$. The kernel of the norm map on divisors has two isomorphic connected components:

$$\text{Nm}^{-1}(\mathcal{O}_C) = P_\eta \cup P_\eta^-,$$

where $P_\eta = (1 - \varphi)J_{C_\eta}^0$ and $P_\eta^- = (1 - \varphi)J_{C_\eta}^1$.

We shall also consider the 2-component subvariety $\text{Nm}^{-1}(\eta) \subset J_{C_\eta}$; this is a torsor over $\text{Nm}^{-1}(\mathcal{O}_C)$ whose components are exchanged under translation by elements of P_η^- . Note that since $\ker f^* = \{\mathcal{O}, \eta\}$ and $f^* \circ \text{Nm} = 1 + \varphi$ it follows that the set of anti-invariant (degree 0) line bundles on C_η is

$$\ker(1 + \varphi) = \text{Nm}^{-1}(\mathcal{O}_C) \cup \text{Nm}^{-1}(\eta).$$

$P_\eta \subset \text{Nm}^{-1}(\mathcal{O}_C)$ is distinguished as the component containing the origin $\mathcal{O} \in J_{C_\eta}$; the component $P_\eta^- \subset \text{Nm}^{-1}(\mathcal{O}_C)$ contains the Prym-canonical curve $C_\eta \xrightarrow{1-\varphi} P_\eta^-$. The components of $\text{Nm}^{-1}(\eta)$, on the other hand, are indistinguishable.

5.1.1 Remark. The $[-1]$ -action on J_{C_η} preserves each of these four components (acting as sheet-involution φ): clearly it permutes the components isomorphically, and so it suffices to note that each component contains fixed points, i.e. points of $J_{C_\eta}[2]$.

For each $\eta \in J_C[2] \setminus \{\mathcal{O}\}$ there exist canonical maps

$$\text{Nm}^{-1}(\eta) \rightarrow \mathcal{SU}_C(2) \rightarrow |2\Theta|$$

where the first map is just direct image $x \mapsto f_*x$; this gives a semistable rank 2 vector bundle of determinant $\det f_*x = \text{Nm}(x) \otimes \det f_*\mathcal{O}_{C_\eta} = \mathcal{O}_C$. This map obviously factors through the sheet-involution, and so the image is a pair of Prym-Kummer varieties. As $\eta \in J_C[2]$ varies, the configuration of these Kummers in $\mathcal{SU}_C(2) \subset |2\Theta|$, including (for $\eta = \mathcal{O}$) the Jacobian Kummer, is called the *Schottky configuration*.

It is usually convenient to view the Kummer varieties of the Schottky configuration as the images of $\text{Nm}^{-1}(\mathcal{O}_C) = P_\eta \cup P_\eta^-$, though this can only be done up to the translation action of $J_C[2]/\eta$. For this we introduce the $J_C[2]$ -torsor

$$S(\eta) = \{\zeta \mid \zeta^2 = \eta\} \subset J_C.$$

Notice that pull-back gives an isomorphism of groups

$$f^* : J_C[2]/\eta \xrightarrow{\sim} \text{Nm}^{-1}(\mathcal{O}_C)[2] := \text{Nm}^{-1}(\mathcal{O}_C) \cap J_{C_\eta}[2],$$

and correspondingly an isomorphism of torsors

$$f^* : S(\eta)/\eta \xrightarrow{\sim} \text{Nm}^{-1}(\eta)[2].$$

Each choice of $\zeta \in S(\eta)/\eta$ determines an identification $\mathrm{Nm}^{-1}(\mathcal{O}_C) \xrightarrow{\sim} \mathrm{Nm}^{-1}(\eta)$ by translation $x \mapsto f^*(\zeta) \otimes x$. This sends 2-torsion points to 2-torsion points, and determines a map

$$\mathrm{Sch}_\zeta : P_\eta \cup P_\eta^- \rightarrow \mathcal{SU}_C(2)$$

by $x \mapsto f_*(f^*(\zeta) \otimes x) = \zeta \otimes f_*x$. The image is independent of the choice of ζ and is precisely the fixed-point set of the involution $\otimes \eta$ of $\mathcal{SU}_C(2)$. Moreover, the linear span of the image of P_η in $|2\Theta|$ can be canonically identified with the linear series $|2\Xi|$ where Ξ is the canonical theta divisor on the dual abelian variety, and represents the principal polarisation on P_η induced from that on $J(C_\eta)$. Thus for each $\zeta \in S(\eta)/\eta$ there is a commutative diagram

$$(19) \quad \begin{array}{ccc} P_\eta & \xrightarrow{\mathrm{Sch}_\zeta} & \mathcal{SU}_C(2) \\ \downarrow & & \downarrow \\ |2\Xi| & \hookrightarrow & |2\Theta|. \end{array}$$

5.2 S_4 -curves over \mathbb{P}^1

When the curve C is trigonal the Pryms of the Schottky configuration are in fact Jacobians, by a well-known construction of Recillas. We shall next review this construction, as formulated by Donagi [3].

To any trigonal curve $C \xrightarrow{3:1} \mathbb{P}^1$ together with a nontrivial 2-torsion point $\eta \in J_C[2]$ one can associate in a natural way a tetragonal curve $R_\eta \xrightarrow{4:1} \mathbb{P}^1$ whose polarised Jacobian is isomorphic to the Prym variety of $C_\eta \rightarrow C$. By definition R_η is one of two isomorphic connected components of the fibre product

$$\begin{array}{ccc} R_\eta \cup R'_\eta & \subset & S^3 C_\eta \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \hookrightarrow & S^3 C \end{array}$$

where $\mathbb{P}^1 \hookrightarrow S^3 C$ is the trigonal pencil, and the right-hand vertical map is the 8:1 cover induced from $C_\eta \rightarrow C$. Conversely $C_\eta = S_{\mathbb{P}^1}^2 R_\eta \subset S^2 R_\eta$ with the obvious involution. In particular, C_η and R_η are in (3,2)-correspondence, and we shall denote the incidence curve by $\mathcal{I} \subset C_\eta \times R_\eta$.

We shall refer to R_η as the *Recillas curve*, and we shall consider its Galois closure $\mathcal{G}(R_\eta) \rightarrow \mathbb{P}^1$ with Galois group S_4 . As in section 4.1 the points of $\mathcal{G}(R_\eta)$ correspond to orderings of the g_4^1 -divisors in R_η .

5.2.1 Notation. We shall view S_4 as the permutation group of $\{0, 1, 2, 3\}$, containing S_3 , the permutations of $\{1, 2, 3\}$, as a subgroup. We shall fix elements $\sigma = (23)$, $\tau = (123)$ (as in earlier sections) and also $\sigma' = (01)$, $\varphi = (0213)$. (Note that φ is uniquely determined up to inverse by the condition $\sigma\sigma' = \varphi^2$. The reason for using the same notation φ as for the sheet-involution of C_η will appear in a moment.) We shall then refer to the subgroups ‘Klein’, generated by the conjugates of $\sigma\sigma' = \varphi^2$, ‘dihedral’ $= \langle \sigma, \varphi \rangle$ and A_4 the alternating subgroup.

S_4 has five irreducible representations and character table:

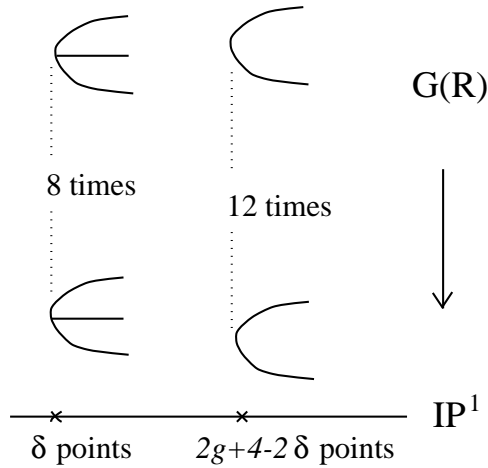
	1	σ	τ	φ	$\sigma\sigma'$
1	1	1	1	1	1
ε	1	-1	1	-1	1
ρ	3	1	0	-1	-1
$\rho \otimes \varepsilon$	3	-1	0	1	-1
2	2	0	-1	0	2

Here ρ is the 3-dimensional reflection representation and **2** is the 2-dimensional reflection representation of $S_4/\text{Klein} \cong S_3$.

The various curves in the Recillas construction are related by the following diagram which extends (17):

$$\begin{array}{ccccc}
 & & \mathcal{G}(R_\eta) & & \\
 & \swarrow & & \searrow & \\
 \mathcal{G}(C) \times_C C_\eta & & & & \mathcal{I} \\
 \downarrow & \searrow & & 2:1 \swarrow & \\
 \mathcal{G}(C) & & C_\eta & & \downarrow 3:1 \\
 \downarrow & \searrow & \downarrow & & \\
 {}^{g+1-\delta}H & & {}^gC & & {}^{g-1}R_\eta \\
 & \searrow & \downarrow & \swarrow & \\
 & & \mathbb{P}^1 & &
 \end{array}
 \tag{20}$$

By comparison of §4.1 and [3] p.74, one sees that the Galois cover $\mathcal{G}(R_\eta) \rightarrow \mathbb{P}^1$ has branching behaviour:



By Riemann-Hurwitz it follows that $\mathcal{G}(R_\eta)$ has genus $12g + 1 - 4\delta$, and in particular we see:

5.2.2 LEMMA. *The 4:1 map $\mathcal{G}(R_\eta) \rightarrow \mathcal{G}(C)$ is unramified.*

Diagram (20) is Galois-dual to the tower of subgroups (and these diagrams coincide with those of [3] p.73):

$$(21) \quad \begin{array}{ccccc} & & 1 & & \\ & \swarrow & & \searrow & \\ \langle \sigma\sigma' \rangle & & & & \langle \sigma \rangle \\ & \downarrow & & \swarrow & \downarrow \\ & \text{Klein} & \langle \sigma, \sigma' \rangle & & \\ & \downarrow & \searrow & \downarrow & \\ & A_4 & \text{dihedral} & & S_3 \\ & & \searrow & \downarrow & \swarrow \\ & & & S_4 & \end{array}$$

Note that the quotient $\text{dihedral}/\langle \sigma, \sigma' \rangle \cong \mathbb{Z}/2$ is generated by φ , which therefore acts as sheet-interchange of C_η over C , consistently with our earlier notation.

5.2.3 PROPOSITION. *As S_4 -modules we have*

$$H^1(\mathcal{G}(R_\eta), \mathcal{O}) = (g + 1 - \delta)\mathbb{C}_\varepsilon \oplus g\mathbb{C}_2^2 \oplus (g - 1)\mathbb{C}_\rho^3 \oplus (2g + 1 - \delta)\mathbb{C}_{\rho \otimes \varepsilon}^3.$$

Proof. One proceeds the same way as in the dimension computation of §4.3. By Atiyah-Bott we have, for 2-cycles $\sigma \in S_4$,

$$\text{trace } \sigma|_{H^1(\mathcal{G}(R_\eta), \mathcal{O})} = 1 - \frac{1}{2}|\text{Fix}(\sigma)| = -2g - 3 + 2\delta;$$

for 3-cycles $\tau \in S_4$,

$$\text{trace } \tau|_{H^1(\mathcal{G}(R_\eta), \mathcal{O})} = 1 - \delta \left\{ \frac{1}{1 - \omega} + \frac{1}{1 - \omega^2} \right\} = 1 - \delta;$$

while the trace of other nontrivial group elements is 1, and the identity element of S_4 has trace $g(\mathcal{G}(R_\eta)) = 12g + 1 - 4\delta$. If the representation on $H^1(\mathcal{G}(R_\eta), \mathcal{O})$ has character $a\mathbf{1} + b\varepsilon + c\rho + d\rho \otimes \varepsilon + e\mathbf{2}$ for integers a, b, c, d, e , then comparing with the character table 5.2.1 we have to solve:

$$\begin{array}{rcllcl} 12g + 1 - 4\delta & = & a & +b & +3c & +3d & +2e \\ -2g - 3 + 2\delta & = & a & -b & +c & -d & \\ 1 - \delta & = & a & +b & & & -e \\ 1 & = & a & -b & -c & +d & \\ 1 & = & a & +b & -c & -d & +2e \end{array}$$

These equations give $(a, b, c, d, e) = (0, g + 1 - \delta, g - 1, 2g + 1 - \delta, g)$. \square

We can now describe the polarised isomorphism $P_\eta \cong J_{R_\eta}$ in this setting. Both abelian varieties embed by pull-back in the Jacobian of the top curve $\mathcal{G}(R_\eta)$; and this Jacobian is acted on by the group-ring $\mathbb{Z}[S_4]$.

5.2.4 PROPOSITION. *The action of $1 + \varphi^2 \in \mathbb{Z}[S_4]$ on the Jacobian $J_{\mathcal{G}(R_\eta)}$ restricts to the polarised isomorphism of Recillas $\psi : J_{R_\eta} \xrightarrow{\sim} P_\eta$. Moreover, $2\psi^{-1} : P_\eta \rightarrow J_{R_\eta}$ is the restriction of $1 + \tau + \tau^2 \in \mathbb{Z}[S_4]$.*

Proof. It is easy to identify the subspaces covering (the pull-backs of) these two abelian varieties in the S_4 -decomposition of Proposition 5.2.3: for J_{R_η} we look for S_3 -invariant vectors, while for P_η we look for $\langle \sigma, \sigma' \rangle$ invariant vectors on which the sheet-involution $\varphi : C_\eta \leftrightarrow C_\eta$ over C acts by -1 . One quickly finds that these are all in the component $(g - 1) \otimes \mathbb{C}_\rho^3$, where

$$\mathbb{C}_\rho^3 = \{\mathbf{x} \in \mathbb{C}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}$$

with S_4 permuting coordinates. The two varieties are covered, respectively, by the subspaces

$$\begin{aligned} H^1(R_\eta, \mathcal{O}) &= \mathbb{C}^{g-1} \otimes \mathbf{l}, & \mathbf{l} &= \tfrac{1}{2}(3, -1, -1, -1) \in \mathbb{C}_\rho^3, \\ H^1_-(C_\eta, \mathcal{O}) &= \mathbb{C}^{g-1} \otimes \mathbf{n}, & \mathbf{n} &= (1, 1, -1, -1) \in \mathbb{C}_\rho^3. \end{aligned}$$

One notes at once (see 5.2.1) that $(1 + \varphi^2)\mathbf{l} = \mathbf{n}$ and $(1 + \tau + \tau^2)\mathbf{n} = 2\mathbf{l}$. This demonstrates the proposition as far as an *isogeny* $\psi : J_{R_\eta} \rightarrow P_\eta$; we have to see that ψ is the Recillas isomorphism as claimed. But the latter, by construction, is induced from the inclusion $C_\eta \subset S^2 R_\eta$, and this fits into a commutative diagram:

$$\begin{array}{ccc} \mathcal{G}(R_\eta) & \xrightarrow{1+\varphi^2} & S^2 \mathcal{G}(R_\eta) \\ \downarrow & & \downarrow \\ C_\eta & \hookrightarrow & S^2 R_\eta \end{array}$$

We conclude that via pull-back, $1 + \varphi^2$ gives the polarised isomorphism $\psi : J_{R_\eta} \xrightarrow{\sim} P_\eta$. \square

5.3 Trigonal Schottky

Given a choice of $\zeta \in S(\eta)/\eta$ we can identify P_η with one component of $\text{Nm}^{-1}(\eta)$. When C is trigonal P_η is canonically identified with the Jacobian J_{R_η} , and the latter (more precisely, its Kummer) can therefore be viewed as a component of the Schottky configuration:

$$\text{Sch}_\zeta : J_{R_\eta} \rightarrow \text{Nm}^{-1}(\eta) \rightarrow \mathcal{SU}_C(2).$$

In Proposition 5.3.2 we shall describe explicitly how rank 2 vector bundles on C are constructed from line bundles on R_η using diagrams (20) and (21); in the next section we will then extend the construction to rank 2 vector bundles on R_η .

Let us denote by π_R the map $\mathcal{G}(R_\eta) \rightarrow R_\eta$; then for any line bundle $L \in J_{R_\eta}$ we consider the collection of bundles on $\mathcal{G}(R_\eta)$ obtained by applying the S_4 -action to $L_0 := \pi_R^* L$. Since L_0 is S_3 -invariant there are just four isomorphism classes, and these are in 1:1 correspondence with the left cosets of $S_3 < S_4$. If we label the bundles as

$$\begin{aligned} L_1 &= \varphi^2 L_0 \\ L_2 &= \varphi L_0 \\ L_3 &= \varphi^3 L_0 \end{aligned}$$

then the Galois action of $S_3 < S_4$ will correspond to the usual permutation action on the subscripts 1, 2, 3 (since $\varphi = (0213)$).

5.3.1 *Remark.* Note that over a divisor $w + x + y + z \in g_4^1 = \mathbb{P}^1$ a point of the Galois curve represents an ordering of w, x, y, z and the fibres of the bundles L_0, L_1, L_2, L_3 over this point are canonically isomorphic, respectively, to the fibres L_w, L_y, L_x, L_z of L .

We now consider the rank 2 vector bundle

$$E'_{L,N} = NL_0L_1 \oplus N^{-1}L_2L_3$$

where N is a line bundle on $\mathcal{G}(R_\eta)$ which will be chosen in a moment. First note that if N is trivial then $E' = E'_{L,\mathcal{O}}$ is invariant under the dihedral group $\langle \sigma, \varphi \rangle$, since $\sigma E' = L_0L_1 \oplus L_3L_2 = E'$ (acting trivially in the fibres at fixed points of σ), and $\varphi E' = L_2L_3 \oplus L_1L_0 = E'$ (with no fixed points of φ to consider since C is smooth). It follows that $E'_{L,\mathcal{O}}$ descends to a bundle on C .

More generally, the same is true of $E'_{L,N}$ provided N is the pull-back of a line bundle in

$$\ker(1 + \varphi) = \text{Nm}^{-1}(\mathcal{O}_C) \cup \text{Nm}^{-1}(\eta) \subset J_{C_\eta},$$

since then N will be invariant under σ and anti-invariant under φ .

On the other hand, it follows from Proposition 5.2.3 that $\pi_R^* J_{R_\eta} \subset \ker(1 + \varphi + \varphi^2 + \varphi^3)$; hence $\det E'_{L,N} = \mathcal{O}_{\mathcal{G}(R_\eta)}$. Consequently the bundle $E_{L,N} \rightarrow C$ which pulls back to $E'_{L,N}$ satisfies

$$\det E_{L,N} \in \{\mathcal{O}_C, \eta\}.$$

It turns out that $\det E_{L,N} = \mathcal{O}_C$ if and only if N comes from $\text{Nm}^{-1}(\eta)$; and that we recover the Schottky map $\text{Sch}_\zeta : J_{R_\eta} \rightarrow \mathcal{SU}_C(2)$ if we choose N to be the pull-back of $\zeta \in \text{Nm}^{-1}(\eta)[2] \cong S(\eta)/\eta$:

5.3.2 PROPOSITION. *For each $\zeta \in S(\eta)/\eta$ the following diagram commutes:*

$$\begin{array}{ccc} & & P_\eta \\ & \nearrow \text{Recillas} & \downarrow \text{Sch}_\zeta \\ J_{R_\eta} & \rightarrow & \mathcal{SU}_C(2) \\ L & \mapsto & E_{L,\zeta} \end{array}$$

where $E_{L,\zeta}$ is defined by $\pi_{C_\eta}^* f^* E_{L,\zeta} = E'_{L,\pi_{C_\eta}^* \circ f^*(\zeta)}$, for the maps $\mathcal{G}(R_\eta) \xrightarrow{\pi_{C_\eta}} C_\eta \xrightarrow{f} C$.

Proof. By Proposition 5.2.4 the Recillas map $J_{R_\eta} \rightarrow P_\eta$ is given, in the Jacobian of $\mathcal{G}(R_\eta)$, by $L \mapsto (1 + \varphi^2)L = L_0L_1$. We shall view the latter as a line bundle on C_η ; we then have to show that its image $\zeta \otimes f_*(L_0L_1) \in \mathcal{SU}_C(2)$ pulls back to $E'_{L, \pi_{C_\eta}^* \circ f^*(\zeta)}$. But this pull-back is

$$\begin{aligned} \pi_{C_\eta}^* f^*(f_*(f^*\zeta \otimes L_0L_1)) &= \pi_{C_\eta}^*((f^*\zeta) \otimes L_0L_1 \oplus \varphi((f^*\zeta) \otimes L_0L_1)) \\ &= \pi_{C_\eta}^*((f^*\zeta) \otimes L_0L_1 \oplus (f^*\zeta^{-1}) \otimes L_2L_3) \\ &= E'_{L, \pi_{C_\eta}^* \circ f^*(\zeta)}. \end{aligned}$$

□

5.4 Nonabelian Schottky

Extending the preceding construction, we can use the diagrams (20) and (21) to construct Galois $\text{Spin}(8)$ -bundles from rank 2 vector bundles on the Recillas curve.

Given a rank 2 vector bundle $E \rightarrow R_\eta$ we consider the bundles on $\mathcal{G}(R_\eta)$ obtained by applying the S_4 -action to $E_0 := \pi_R^*E$. Again, S_3 -invariance implies that there are just four isomorphism classes in 1:1 correspondence with the left cosets of $S_3 < S_4$. We write

$$\begin{aligned} E_1 &= \varphi^2 E_0 \\ E_2 &= \varphi E_0 \\ E_3 &= \varphi^3 E_0. \end{aligned}$$

The Galois action of S_4 on the the bundles E_0, E_1, E_2, E_3 then coincides with permutation of the indices.

5.4.1 LEMMA-DEFINITION. *Fix an element $\zeta \in S(\eta)/\eta$, and denote its pull-back to $\mathcal{G}(R_\eta)$ by $\zeta' \in J_{\mathcal{G}(R_\eta)}[2]$. Given a semistable rank 2 vector bundle $E \rightarrow R_\eta$ with $\det E = \mathcal{O}$, the rank 8 orthogonal bundles (where $E_0E_1 = E_0 \otimes E_1$ etc)*

$$\begin{aligned} V_{E, \zeta} &= \zeta' \otimes (E_0E_1 \oplus E_2E_3), \\ S_{E, \zeta}^+ &= \tau\zeta' \otimes (E_0E_2 \oplus E_3E_1), \\ S_{E, \zeta}^- &= \tau^2\zeta' \otimes (E_0E_3 \oplus E_1E_2), \end{aligned}$$

are semistable and descend to a Galois $\text{Spin}(8)$ -bundle $F_{E, \zeta} \in \mathcal{N}_C$.

Proof. By Proposition 2.1.1 these three bundles are the standard representations of a $\text{Spin}(8)$ -bundle, and by construction they are permuted isomorphically by the 3-cycle $\tau \in S_4$ under the Galois action. The involution σ , on the other hand, lifts to an involution of V_F with 1-dimensional -1 -eigenspace $\bigwedge^2(E_2)_x$ at each fixed point $x \in \mathcal{G}(R_\eta)$. (So the dihedral invariance and descent to the curve C of the previous section *fail* in this rank 2 case.) However, the triple is invariant under the Klein subgroup and by Lemma 5.2.2 the map $\mathcal{G}(R_\eta) \rightarrow \mathcal{G}(C)$ is unramified. It follows that the bundles descend to $\mathcal{G}(C)$ where they satisfy the conditions of Proposition 2.2.2. □

5.4.2 *Remark.* Observe that if we define $R_0 = C \cup \mathbb{P}^1$ and extend any rank 2 bundle $E \rightarrow C$ to the trivial bundle on the component \mathbb{P}^1 , then the above construction reduces to that of Lemma 2.3.1.

We shall write $\text{Sch}_\zeta : \mathcal{SU}_{R_\eta}(2) \rightarrow \mathcal{N}_C$ for the map determined by 5.4.1, and verify that this is consistent with the notation for Pryms:

5.4.3 **THEOREM.** *For each $\eta \in J_C[2]$ and choice of $\zeta \in S(\eta)/\eta$ the following diagram commutes:*

$$\begin{array}{ccc} P_\eta & \xrightarrow{\sim} J_{R_\eta} \rightarrow & \mathcal{SU}_{R_\eta}(2) \\ \text{Sch}_\zeta \downarrow & & \downarrow \text{Sch}_\zeta \\ \mathcal{SU}_C(2) & \longrightarrow & \mathcal{N}_C \end{array}$$

Proof. Let us fix the following notation for the various maps which we will need:

$$\begin{array}{ccccc} & & \mathcal{G}(R_\eta) & & \\ & \swarrow \pi_{\mathcal{G}(C)} & & \searrow \pi_{C_\eta} & \\ \mathcal{G}(C) & & & & C_\eta \\ & \searrow \pi_C & & \swarrow f & \\ & & C & & \end{array}$$

Pick line bundles $L \in J_{R_\eta}$ and $N \in P_\eta$ corresponding under the Recillas isomorphism. L gives rise to a Galois bundle $F_{L,\zeta} = (V_{L,\zeta}, S_{L,\zeta}^+, S_{L,\zeta}^-)$ on $\mathcal{G}(C)$ (via the right-hand side of the diagram) which pulls back to the triple of orthogonal bundles on $\mathcal{G}(R_\eta)$ (by 5.4.1):

$$(22) \quad \begin{aligned} \pi_{\mathcal{G}(C)}^* V_{L,\zeta} &= \zeta' \otimes \{(L \oplus L^{-1})(\varphi^2 L \oplus \varphi^2 L^{-1}) \oplus (\varphi^3 L \oplus \varphi^3 L^{-1})(\varphi L \oplus \varphi L^{-1})\}, \\ \pi_{\mathcal{G}(C)}^* S_{L,\zeta}^+ &= \tau \zeta' \otimes \{(L \oplus L^{-1})(\varphi L \oplus \varphi L^{-1}) \oplus (\varphi^2 L \oplus \varphi^2 L^{-1})(\varphi^3 L \oplus \varphi^3 L^{-1})\}, \\ \pi_{\mathcal{G}(C)}^* S_{L,\zeta}^- &= \tau^2 \zeta' \otimes \{(L \oplus L^{-1})(\varphi^3 L \oplus \varphi^3 L^{-1}) \oplus (\varphi L \oplus \varphi L^{-1})(\varphi^2 L \oplus \varphi^2 L^{-1})\}. \end{aligned}$$

Via the left-hand side of the diagram, on the other hand, N gives rise to a Galois bundle $F_{N,\zeta} \in \mathcal{N}_C$ represented by a triple $(V_{N,\zeta}, S_{N,\zeta}^+, S_{N,\zeta}^-)$ and we shall show that these bundles split under pull-back to $\mathcal{G}(R_\eta)$ into the same line bundle summands as (22).

Starting with $N \in P_\eta$ we construct $E_{N,\zeta} = \zeta \otimes f_* N \rightarrow C$; then $V_{N,\zeta} = \mathbb{C}^2 \otimes \pi_C^* E_{N,\zeta} \oplus \tau(\pi_C^* E_{N,\zeta}) \otimes \tau^2(\pi_C^* E_{N,\zeta})$ etc. Now

$$\begin{aligned} \pi_{\mathcal{G}(C)}^*(\pi_C^* E_{N,\zeta}) &= \pi_{C_\eta}^* f^*(\zeta \otimes f_* N) \\ &= \zeta' \otimes \pi_{C_\eta}^*(N \oplus \varphi(N)) \\ &= \zeta' \otimes \pi_{C_\eta}^*(N \oplus N^{-1}), \end{aligned}$$

where we use the fact that N is anti-invariant under the sheet-involution $\varphi : C_\eta \leftrightarrow C_\eta$. It follows (denoting $\pi_{C_\eta}^* N$ by N for simplicity, and using the fact that $\zeta' \otimes \tau \zeta' \otimes \tau^2 \zeta' = \mathcal{O}$) that

$$\begin{aligned}
(23) \quad \pi_{\mathcal{G}(C)}^* V_{N,\zeta} &= \zeta' \otimes \{ 2N \oplus 2N^{-1} \oplus (\tau N \oplus \tau N^{-1})(\tau^2 N \oplus \tau^2 N^{-1}) \}, \\
\pi_{\mathcal{G}(C)}^* S_{N,\zeta}^+ &= \tau \zeta' \otimes \{ 2\tau N \oplus 2\tau N^{-1} \oplus (\tau^2 N \oplus \tau^2 N^{-1})(N \oplus N^{-1}) \}, \\
\pi_{\mathcal{G}(C)}^* S_{N,\zeta}^- &= \tau^2 \zeta' \otimes \{ 2\tau^2 N \oplus 2\tau^2 N^{-1} \oplus (N \oplus N^{-1})(\tau N \oplus \tau N^{-1}) \}.
\end{aligned}$$

For each of the bundles $V_{L,\zeta}, V_{N,\zeta}$ we expand the expression in (22) and (23). Each line bundle summand is represented by a vector in $\mathbb{C}^{g-1} \otimes \mathbb{C}_\rho^3 \subset H^1(\mathcal{G}(R_\eta), \mathcal{O})$. (See the proof of Proposition 5.2.4.) By hypothesis L, N are represented respectively by $\mathbf{x} \otimes \mathbf{l}, \mathbf{x} \otimes \mathbf{n}$ for some $\mathbf{x} \in \mathbb{C}^{g-1}$, where $\mathbf{l} = \frac{1}{2}(3, -1, -1, -1)$ and $\mathbf{n} = (1, 1, -1, -1)$. We can therefore identify each summand, as a vector in this representation, using the S_4 -action. For example, φL is given by the vector $\mathbf{x} \otimes \varphi \mathbf{l} = \mathbf{x} \otimes (-1, -1, 3, -1)$, and $L \otimes \varphi L$ by $\mathbf{x} \otimes (\mathbf{l} + \varphi \mathbf{l}) = \mathbf{x} \otimes (2, -2, 2, -2)$. Computing in this way, we find that the summands of $\pi_{\mathcal{G}(C)}^* V_{L,\zeta}$ are \mathbf{x} tensored with the vectors:

$$\begin{aligned}
&\pm(1, 1, -1, -1), \\
&\pm(1, 1, -1, -1), \\
&\pm(2, -2, 0, 0), \\
&\pm(0, 0, 2, -2).
\end{aligned}$$

We now play the same game with $\pi_{\mathcal{G}(C)}^* V_{N,\zeta}$, representing N by $\mathbf{x} \otimes (1, 1, -1, -1)$, τN by $\mathbf{x} \otimes (1, -1, 1, -1)$ etc. After computing we find that $\pi_{\mathcal{G}(C)}^* V_{N,\zeta}$ is represented by *exactly the same eight vectors* as above. We conclude that $\pi_{\mathcal{G}(C)}^* V_{L,\zeta} \cong \pi_{\mathcal{G}(C)}^* V_{N,\zeta}$, and hence by construction $V_{L,\zeta} \cong V_{N,\zeta}$. Repeating this calculation for the spinor bundles shows likewise that $S_{L,\zeta}^\pm \cong S_{N,\zeta}^\pm$, and hence $F_{L,\zeta} = F_{N,\zeta}$. \square

5.4.4 *Remark.* There is a natural ‘Heisenberg’ action of $J_C[2]$ on the space \mathcal{N}_C : for $\eta \in J_C[2]$ we define

$$\begin{aligned}
V_F &\mapsto \pi_C^* \eta \otimes V_F, \\
S_F^+ &\mapsto \tau(\pi_C^* \eta) \otimes S_F^+, \\
S_F^- &\mapsto \tau^2(\pi_C^* \eta) \otimes S_F^-.
\end{aligned}$$

The three image bundles are cyclically permuted by τ , and the last two exchanged by σ , and so they define a Galois $\text{Spin}(8)$ -bundle $\eta \cdot F \in \mathcal{N}_C$. Moreover, one readily checks, using the relation $\pi_C^* \eta \otimes \tau(\pi_C^* \eta) \otimes \tau^2(\pi_C^* \eta) = \mathcal{O}$, that the map $\mathcal{SU}_C(2) \hookrightarrow \mathcal{N}_C$ is equivariant where $J_C[2]$ acts by tensor product on rank 2 vector bundles. We expect that the fixed-point set of this involution is precisely the ‘Schottky locus’ $\mathcal{SU}_{R_\eta}(2) \cup \mathcal{SU}_{R'_\eta}(2)$.

References

- [1] M.F. ATIYAH, R. BOTT: A Lefschetz fixed point formula for elliptic complexes II, Ann. Math. 88 (1968) 451–491;
- [2] S. BRIVIO, A. VERRA: The theta divisor of $\mathcal{SU}_C(2, 2d)^s$ is very ample if C is not hyper-elliptic, Duke Math. J. 82 no.3 (1996) 503–552;

- [3] R. DONAGI: The fibres of the Prym map, in *Curves, Jacobians and Abelian Varieties*, Contemporary Math. 136 (1992) 55–125;
- [4] C.C. CHEVALLEY: *The Algebraic Theory of Spinors*, Columbia University Press 1954;
- [5] A.J. FEINGOLD, I.B. FRENKEL, J.F.X. RIES: Spinor construction of vertex operator algebras, triality and $E_8^{(1)}$, Contemporary Math. 121 (1991);
- [6] B. VAN GEEMEN, E. IZADI: The tangent space to the moduli space of vector bundles on a curve and the singular locus of the theta divisor of the Jacobian, to appear;
- [7] B. VAN GEEMEN, E. PREVIATO: Prym varieties and the Verlinde formula, Math. Ann. 294 (1992) 741–754;
- [8] F.R. HARVEY: *Spinors and calibrations*, Perspectives in Math. 9, Academic Press 1990;
- [9] S. KUMAR, M.S. NARASIMHAN, A. RAMANATHAN: Infinite Grassmannian and moduli space of G -bundles, Math. Ann. 300 (1993) 395–423;
- [10] S. MACLANE: *Homology*, Springer Grundlehren 114 (1963);
- [11] M.S. NARASIMHAN, S. RAMANAN: Moduli of vector bundles over a compact Riemann surface, Ann. of Math. 89 (1969) 19–51;
- [12] M.S. NARASIMHAN, S. RAMANAN: Generalised Prym varieties as fixed points, J. Indian Math. Soc. 39 (1975) 1–19;
- [13] M.S. NARASIMHAN, S. RAMANAN: 2θ -linear systems on abelian varieties, in *Vector Bundles on Algebraic Varieties*, Tata Institute, Bombay 1984;
- [14] W.M. OXBURY: Prym varieties and the moduli of spin bundles, in *Algebraic Geometry*, ed. P.E. Newstead, Lect. Notes in Pure App. Math. vol 200 (Marcel Dekker 1998) 351–376;
- [15] W.M. OXBURY, C. PAULY: Heisenberg invariant quartics and $SU_C(2)$ for a curve of genus four, Math. Proc. Camb. Phil. Soc. 125 (1999) 295–319;
- [16] S. RAMANAN: Orthogonal and spin bundles over hyperelliptic curves, in *Geometry and Analysis, papers dedicated to V.K. Patodi*, Springer 1981;
- [17] A. RAMANATHAN: Stable principal bundles on a compact Riemann surface, Math. Ann. 213 (1975) 129–152;
- [18] S. RECILLAS: Jacobians of curves with a g_4^1 are Prym varieties of trigonal curves, Bol. Soc. Mat. Mexicana 19 (1974) 9–13.

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